Some Symmetric Curvature Conditions on Kenmotsu Manifolds

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Abstract: *In this paper, we study locally and globally φ-symmetric Kenmotsu manifolds. In both curvature conditions, it is proved that the manifold is of constant negative curvature - 1 and globally φ-Weyl projectively symmetric Kenmotsu manifold is an Einstein manifold. Finally, we give an example of 3-dimensional Kenmotsu manifold.*

Keywords: Kenmotsu manifold, Locally φ-symmetric, Globally φ-symmetric, Weyl projective curvature tensor

1 Introduction

Tanno [8] classified connected almost contact metric manifolds whose automorphism groups have the maximum dimension. For such a manifold M, the sectional curvature of plane sections containing ξ is a constant, say c, where ξ is a global vector field (or contravariant vector field or Reeb vector field). If $c > 0, M$ is a homogeneous Sasakian manifold of constant φ-sectional curvature. If $c = 0, M$ is the product of a line or circle with a Kaehler manifold of constant holomorphic curvature. If $c < 0, M$ is a warped product space $R \times_f C^n$. In [5], Kenmotsu abstracted the differential geometric properties of the case if $c < 0$ and also introduced the notion of a class of almost contact Riemannian manifolds with some special conditions. We call this type of manifold, a Kenmotsu manifold.

Takahashi [7] introduced the notion of locally φ-symmetric Sasakian manifold as a weaker version of local symmetry of such manifold. In this paper, we study locally φ-symmetric Kenmotsu manifold, globally φ-symmetric Kenmotsu manifold and globally φ-Weyl projectively symmetric Kenmotsu manifold. In first two cases, we have obtained the result that the manifold is of constant negative curvature - 1. In next condition, it is shown that the manifold is an Einstein manifold with scalar curvature $r = n(n-1)$.

2 Preliminaries

Let M be an n-dimensional (where $n = 2m+1$) almost contact manifold with an almost contact metric structure (φ, ξ, η, g), where φ is a (1, 1) tensor field, ξ is a Reeb vector field (or contravariant vector field), η is a 1-form and g is a compatible Riemannian metric such that

$$\varphi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \varphi\xi = 0, \eta(\varphi X) = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \xi) = \eta(X) \tag{2.3}$$

for all $X, Y \in T(M)$ [1, 2]. An almost contact metric manifold (M^n, g) is said to be a Kenmotsu manifold if the conditions

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{2.4}$$

$$\nabla_X \xi = X - \eta(X)\xi \tag{2.5}$$

hold in M, where ∇ is the Levi-Civita connection of g [5].
In an n-dimensional ($n = 2m+1$) Kenmotsu manifold, the following relations hold [5]

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{2.6}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.8}$$

$$S(\varphi X,\varphi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{2.9}$$

for any vector fields X, Y on M, where R and S are the Riemannian curvature tensor and the Ricci tensor respectively.

Definition 2.1. A Kenmotsu manifold (M^n, g) is said to be a locally φ-symmetric manifold if the condition

$$\varphi^2((\nabla_W R)(X,Y)Z) = 0 \tag{2.10}$$

holds for any vector fields X, Y, Z, W orthogonal to ξ, that is for any horizontal vector fields X, Y, Z, W.

This notion was introduced by Takahashi [7] for Sasakian manifold.

Definition 2.2. An n-dimensional Kenmotsu manifold M is said to be globally φ-symmetric if it satisfies the condition

$$\varphi^2((\nabla_W R)(X,Y)Z) = 0 \tag{2.11}$$

for arbitrary vector fields X, Y, Z and W on M.

The Weyl projective curvature tensor P of type (1, 3) on a Riemannian manifold (M^n, g) is defined by [3]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] \tag{2.12}$$

for any $X, Y, Z \in \chi(M)$, the set of vector fields.

Definition 2.3. A Kenmotsu manifold M of dimension n is said to be globally φ-Weyl projectively symmetric if the Weyl projective curvature tensor P satisfies

$$\varphi^2((\nabla_W P)(X,Y)Z) = 0 \tag{2.13}$$

for all vector fields $X, Y, Z, W \in \chi(M)$.

3 Results and Discussions

Theorem 3.1. *A Kenmotsu manifold (M^n, g) is locally φ-symmetric if and only if*

$$(\nabla_W R)(X,Y)Z = g(R(X,Y)Z, W)\xi$$

for any horizontal vector fields X, Y, Z and W.

Proof. Let us consider an n-dimensional Kenmotsu manifold which satisfies the condition (2.10). Then by the use of (2.1), the relation (2.10) yields

$$(\nabla_W R)(X,Y)Z + g((\nabla_W R)(X,Y)\xi, Z)\xi = 0. \tag{3.1}$$

From (2.7), we have

$$(\nabla_W R)(X,Y)\xi = (\nabla_W \eta)(X)Y + \eta(X)\nabla_W Y - (\nabla_W \eta)(Y)X - \eta(Y)\nabla_W X. \tag{3.2}$$

In view of (2.6) and (3.2), we obtain

$$\begin{aligned}(\nabla_W R)(X,Y)\xi = {} & g(X,W)Y - g(Y,W)X - \eta(X)\eta(W)Y \\ & + \eta(X)\nabla_W Y + \eta(Y)\eta(W)X - \eta(Y)\nabla_W X.\end{aligned} \tag{3.3}$$

For horizontal vectors X, Y, W, the relation (3.3) reduces to

$$(\nabla_W R)(X,Y)\xi = g(X,W)Y - g(Y,W)X. \tag{3.4}$$

Using (3.4) in the relation (3.1), we get

$$(\nabla_W R)(X,Y)Z) + g(g(X,W)Y - g(Y,W)X, Z)\xi = 0$$

or,

$$(\nabla_W R)(X,Y)Z - g(R(X,Y)Z, W)\xi = 0$$

this implies

$$(\nabla_W R)(X,Y)Z = g(R(X,Y)Z, W)\xi \tag{3.5}$$

for any horizontal vector fields X, Y, Z and W. Next, if the relation (3.5) holds for any vector fields X, Y, Z, W orthogonal to ξ, it follows from $\varphi\xi = 0$ that (2.10) holds and hence the manifold is locally φ-symmetric. This completes the proof of the theorem. □

From (3.5), it also follows that if $(\nabla_W R)(X,Y)Z = 0$, then $R(X,Y)Z = 0$ since W and ξ are non-zero. Thus, we have a corollary

Corollary 3.2. *If an n-dimensional Kenmotsu manifold is locally symmetric, then the manifold is flat.*

Again, in corollary 6 of proposition 5 Kenmotsu [5] proved that if a Kenmotsu manifold is locally symmetric, then it is of constant negative curvature - 1.

Theorem 3.3. *Let M be an n-dimensional Kenmotsu manifold. If M is globally φ-symmetric, then it is locally symmetric.*

Proof. Let M be an n-dimensional Kenmotsu manifold. Suppose that the condition (2.11) holds. Then from (2.1) and (2.11) we obtain

$$-(\nabla_W R)(X,Y)Z + \eta((\nabla_R R)(X,Y)Z)\xi = 0 \tag{3.6}$$

or,

$$(\nabla_W R)(X,Y)Z + g((\nabla_W R)(X,Y)\xi, Z)\xi = 0 \tag{3.7}$$

By the use of equation (3.3) of proposition 5 of [5] as

$$(\nabla_Z R)(X,Y)\xi = g(Z,X)Y - g(Z,Y)X - R(X,Y)Z$$

in the relation (3.7), we get

$$(\nabla_W R)(X,Y)Z + g(X,W)g(Y,Z)\xi - g(Y,W)g(X,Z)\xi - g(R(X,Y)W), Z)\xi = 0. \tag{3.8}$$

In view of (2.7), relation (3.8) reduces to

$$(\nabla_W R)(X,Y)Z = 0. \tag{3.9}$$

Hence the theorem is proved. □

From Theorem 3.3 and corollary 6 of [5], we can state next theorem

Theorem 3.4. *If an n-dimensional Kenmotsu manifold M is globally φ-symmetric, then it is of constant negative curvature - 1.*

Theorem 3.5. *Let (M^n, g) be a Kenmotsu manifold. If M is globally φ-Weyl projectively symmetric, then it is an Einstein manifold with scalar curvature $r = n(n-1)$.*

Proof. Let us consider M is a globally φ-Weyl projectively symmetric manifold. Then (2.13) holds. Now, using (2.1), we obtain

$$-(\nabla_W P)(X,Y)Z + \eta((\nabla_W P)(X,Y)Z)\xi = 0. \tag{3.10}$$

Differentiating(2.12) covariantly with respect to W, we get

$$(\nabla_W P)(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{1}{n-1}[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y)]. \tag{3.11}$$

In view of (3.10) and (3.11), we get

$$\begin{aligned} 0 = &-g((\nabla_W R)(X,Y)Z,U) + \frac{1}{n-1}[(\nabla_W S)(Y,Z)g(X,U) - (\nabla_W S)(X,Z)g(Y,U)] \\ &+ \eta((\nabla_W R)(X,Y)Z)\eta(U) - \frac{1}{n-1}[(\nabla_W S)(Y,Z)\eta(X) - (\nabla_W S)(X,Z)\eta(Y)]\eta(U). \end{aligned} \tag{3.12}$$

Let $\{e_i\}, i = 1,2,...,n$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = U = e_i$, in (3.12) and summing over $i, 1 \leq i \leq n$, we get

$$\begin{aligned} 0 = &-(\nabla_W S)(Y,Z) + \frac{1}{n-1}[n(\nabla_W S)(Y,Z) - (\nabla_W S)(Y,Z)] + \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) \\ &- \frac{1}{n-1}[(\nabla_W S)(Y,Z) - (\nabla_W S)(Z,\xi)\eta(Y)] \end{aligned}$$

or,

$$0 = \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) - \frac{1}{n-1}[(\nabla_W S)(Y,Z) - (\nabla_W S)(Z,\xi)\eta(Y)]. \tag{3.13}$$

Putting $Z = \xi$ in (3.13), we obtain

$$0 = \eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) - \frac{1}{n-1}(\nabla_W S)(Y,\xi) + \frac{1}{n-1}(\nabla_W S)(\xi,\xi)\eta(Y). \tag{3.14}$$

Now, we have

$$\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) = g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi). \tag{3.15}$$

Again, we get

$$\begin{aligned} g(\nabla_W R)(e_i,Y)\xi,\xi) &= g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi) \\ &\quad - g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi) \end{aligned}$$

Since $\{e_i\}$ is an orthonormal basis $\nabla_W e_i = 0$. From (2.7) we have

$$\begin{aligned} g(R(e_i,\nabla_W Y)\xi,\xi) &= g(\eta(e_i)\nabla_W Y - \eta(\nabla_W Y)e_i,\xi) \\ &= \eta(e_i)\eta(\nabla_W Y) - \eta(\nabla_W Y)\eta(e_i) \\ &= 0. \end{aligned}$$

We know that if R is the Riemannian curvature tensor of a Riemannian manifold (M,g) [3, 4, 6], we have

$$g(R(X,Y)Z,U) = -g(R(Z,U)Y,X).$$

Thus, $g(R(e_i,Y)\xi,\xi) + g(R(\xi,\xi)Y,e_i) = 0$ and we get

$$g(\nabla_W R(e_i,Y)\xi,\xi) + g(R(e_i,Y)\xi,\nabla_W \xi) = 0.$$

Using above relations in (3.15), we obtain $g((\nabla_W R)(e_i,Y)\xi,\xi)\eta(e_i) = 0$ and the equation (3.14) reduces to

$$(\nabla_W S)(Y,\xi) - (\nabla_W S)(\xi,\xi)\eta(Y) = 0. \tag{3.16}$$

Now, we have

$$\begin{aligned}(\nabla_W S)(Y,\xi) &= \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi)\\ &= -(n-1)\nabla_W \eta(Y) + (n-1)\eta(\nabla_W Y) - S(Y,W) + \eta(W)S(Y,\xi)\\ &= -(n-1)(\nabla_W \eta)(Y) - S(Y,W) - (n-1)\eta(Y)\eta(W)\\ &= -(n-1)\{g(W,Y) - \eta(Y)\eta(W)\} - S(Y,W) - (n-1)\eta(W)\eta(Y)\\ &= -S(Y,W) - (n-1)g(W,Y).\end{aligned}$$

So, putting $Y = \xi$ in above relation, we get

$$(\nabla_W S)(\xi,\xi) = 0.$$

Using above two relations in (3.16), we obtain

$$S(W,Y) = (n-1)g(W,Y). \tag{3.17}$$

Now, taking an orthonormal frame field at any point of the manifold and contracting over W and Y in (3.17), we get

$$r = n(n-1) \tag{3.18}$$

where r is the scalar curvature.
In view of (3.17) and (3.18), the theorem is proved. □

4 Example of 3-dimensional Kenmotsu Manifold

Let us consider 3-dimensional manifold $M = \{(x,y,z) \in R^3\}$, $z \neq 0$ where (x,y,z) are the standard coordinates of R^3. Let $\{E_1, E_2, E_3\}$ be a linearly independent global frame on M defined by

$$E_1 = z\frac{\partial}{\partial x}, E_2 = z\frac{\partial}{\partial y}, E_3 = -z\frac{\partial}{\partial z}.$$

Let g be a Riemannian metric defined by

$$g(E_1,E_2) = g(E_2,E_3) = g(E_1,E_3) = 0,$$
$$g(E_1,E_1) = g(E_2,E_2) = g(e_3,E_3) = 1.$$

Let η be a 1-form defined by $\eta(U) = g(U,E_3)$ for any $U \in \chi(M)$, the set of vector fields. Let φ be the (1, 1) tensor field defined by

$$\varphi(E_1) = -E_2, \varphi(E_2) = E_1, \varphi(E_3) = 0.$$

Then, using the linearity of φ and g, we have

$$\eta(E_3) = 1, \varphi^2(U) = -U + \eta(U)E_3, g(\varphi U, \varphi V) = g(U,V) - \eta(U)\eta(V),$$

for any vector fields $U, V \in \chi(M)$. Thus, for $E_3 = \xi$, (φ, ξ, η, g) defines an almost contact metric structure on M. Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g. Then, by the definition of Lie bracket, we have

$$\begin{aligned}[E_1,E_3] &= E_1E_3 - E_3E_1\\ &= z\frac{\partial}{\partial x}\Big(-z\frac{\partial}{\partial z}\Big) - \Big(-z\frac{\partial}{\partial z}\Big)\Big(z\frac{\partial}{\partial x}\Big)\\ &= -z^2\frac{\partial^2}{\partial x\partial z} + z\Big(z\frac{\partial^2}{\partial z\partial x} + \frac{\partial}{\partial x}\times 1\Big)\\ &= z\frac{\partial}{\partial x}\\ &= E_1.\end{aligned}$$

Similarly, we obtain $[E_1, E_2] = 0$ and $[E_2, E_3] = E_2$.
Now, we have Koszul formula

$$\begin{aligned}2g(\nabla_U V, W) &= Ug(V, W) + Vg(W, U) - Wg(U, V)\\ &\quad + g([U, V], W) - g([V, W], U) + g([W, U], V).\end{aligned}$$

Using above Koszul formula, we obtain

$$\begin{aligned}2g(\nabla_{E_1} E_3, E_1) &= E_1 g(E_3, E_1) + E_3 g(E_1, E_3) - E_1 g(E_1, E_3)\\ &\quad + g([E_1, E_3], E_1) - g([E_3, E_1], E_1) + g([E_1, E_1], E_3)\\ &= 2g(E_1, E_1).\end{aligned}$$

Similarly, we can calculate

$$2g(\nabla_{E_1} E_3, E_2) = 0 = 2g(E_1, E_2) \quad \text{and} \quad 2g(\nabla_{E_1} E_3, E_3) = 0 = 2g(E_1, E_3).$$

Thus, $g(\nabla_{E_1} E_3, X) = g(E_1, X)$ for all $X \in \chi(M)$.
Therefore, $\nabla_{E_1} E_3 = E_1$.
Proceeding continuously in this way, we obtain

$$\begin{aligned}&\nabla_{E_1} E_3 = E_1, \nabla_{E_1} E_2 = 0, \nabla_{E_1} E_1 = -E_3,\\ &\nabla_{E_2} E_3 = E_2, \nabla_{E_2} E_2 = -E_3, \nabla_{E_2} E_1 = 0,\\ &\text{and } \nabla_{E_3} E_1 = \nabla_{E_3} E_2 = \nabla_{E_3} E_3 = 0.\end{aligned}$$

Now, we get

$$\begin{aligned}&\nabla_{E_1} E_3 = E_1 = E_1 - g(E_1, E_3)E_3,\\ &\nabla_{E_2} E_3 = E_2 = E_2 - g(E_2, E_3)E_3,\\ &\text{and } \nabla_{E_3} E_3 = 0 = E_3 - g(E_3, E_3)E_3.\end{aligned}$$

For $E_3 = \xi$, above results become

$$\nabla_X \xi = X - g(X, \xi)\xi = X - \eta(X)\xi.$$

Thus the second condition (2.5) for Kenmotsu manifold is satisfied. Again, we have

$$(\nabla_{E_1}\varphi)E_1 = \nabla_{E_1}\varphi E_1 - \varphi\nabla_{E_1} E_1 = \nabla_{E_1}(-E_2) - \varphi(-E_3) = 0$$

and

$$g(\varphi E_1, E_1)E_3 - g(E_1, E_3)\varphi E_1 = g(-E_2, E_1) = 0$$

Therefore, we get

$$(\nabla_{E_1}\varphi)E_1 = g(\varphi E_1, E_1) - g(E_1, E_3)\varphi E_1 = 0.$$

Similarly, we can verify other results. Hence we have

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \quad \text{for} \quad E_3 = \xi.$$

Thus, the first condition (2.4) for Kenmotsu manifold is also satisfied. Satisfying two conditions (2.4) and (2.5) for Kenmotsu manifold, the manifold under consideration is a 3-dimensional Kenmotsu manifold.
By the definition of Riemannian curvature tensor in terms of ∇, we have

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Thus, using definition of R, we have

$$\begin{aligned}R(E_1, E_2)E_3 &= \nabla_{E_1}\nabla_{E_2} E_3 - \nabla_{E_2}\nabla_{E_1} E_3 - \nabla_{[E_1, E_2]} E_3\\ &= \nabla_{E_1} E_2 - \nabla_{E_2} E_1\\ &= 0.\end{aligned}$$

Similarly, we obtain

$$R(E_2, E_3)E_3 = -E_2, R(E_1, E_3)E_3 = -E_1, R(E_1, E_2)E_2 = -E_1,$$
$$R(E_2, E_3)E_2 = E_3, R(E_1, E_3)E_2 = 0, R(E_1, E_2)E_1 = E_2,$$
$$R(E_2, E_3)E_1 = 0, R(E_1, E_3)E_1 = E_3.$$

From above curvature relations, it follows that $\varphi^2((\nabla_W R)(X, Y)Z) = 0$. Hence 3-dimensional Kenmotsu manifold is locally φ-symmetric.

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