$\beta-$Change of Riemannian Space and Special Cases of one-form β

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Abstract: *In this paper, we consider $\beta-$change of Finsler metric L given by $\overline{L} = f(L, \beta)$, where f is any positively homogeneous function of degree one in L and β. The objectives of the paper are to obtain the $\beta-$change of a Riemannian space, projective change of Finsler metric and to discuss the special cases of one-form β.*

Keywords: $\beta-$change, Riemannian space, Finsler metric

1 Introduction

Let $F^n = (M^n, L)$ be an $n-$dimensional Finsler space on a differentiable manifold M^n, equipped with the fundamental function $L(x, y)$. In 2012, Shukla et al. [8] introduced the transformation of Finsler metric called Matsumoto change of metric given by

$$\overline{L}(x, y) = \frac{L^2}{L - \beta}$$

where $\beta(x, y) = b_i(x)y^i$ is a one-form on M^n. They obtained the necessary and sufficient condition for this change of Finsler metric to be projective change.

Prasad et al. [7] introduced an exponential change of Finsler metric given by

$$\overline{L}(x, y) = Le^{\beta/L}$$

and they dealt with the imbedding class numbers of the tangent Riemannian space of corresponding spaces.

Recently, Prasad and Kumari [6] introduced the $\beta-$change of Finsler space given by

$$\overline{L}(x, y) = f(L, \beta), \tag{1.1}$$

where f is positively homogeneous function of y^i of degree one in L and β. They also dealt with the imbedding classes of the tangent Riemannian spaces. We have considered the same $\beta-$change given by the equation (1.1) and obtained the necessary and sufficient condition under which this change becomes a projective change. The particular cases when the vector field b_i in β is special one have been discussed. The Berwald's connection coefficients for the $\beta-$changed space have been calculated.

2 Preliminaries

Let $F^n = (M^n, L)$ be a Finsler space equipped with the fundamental function $L(x, y)$ on the smooth manifold M^n. Let $\beta = b_i(x)y^i$ be a one-form on the manifold M^n, then $L \to f(L, \beta)$ is the $\beta-$change of Finsler metric. If we write $\overline{L} = f(L, \beta)$, where f is any positively homogeneous function of degree one in L and β and $\overline{F}^n = (M^n, \overline{L})$, then the Finsler space $\overline{F}^n$ is said to be obtained from F^n by $\beta-$change. The quantities corresponding to $\overline{F}^n$ are denoted by putting bar on those quantities.

The fundamental metric tensor g_{ij}, the normalized element of support l_i and angular metric tensor h_{ij} of F^n are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \qquad l_i = \frac{\partial L}{\partial y^i} \qquad \text{and} \qquad h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j} = g_{ij} - l_i l_j.$$

We shall denote the partial derivative with respect to x^i and y^i by ∂_i and $\dot{\partial}_i$ respectively and write

$$L_i = \dot{\partial}_i L, \qquad L_{ij} = \dot{\partial}_i \dot{\partial}_j L, \qquad L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L.$$

Then,

$$L_i = l_i, \qquad L^{-1} h_{ij} = L_{ij}.$$

The geodesic of F^n are given by the system of differential equations

$$\frac{d^2 x^i}{ds^2} + 2G^i \left(x, \frac{dx}{ds} \right) = 0,$$

where $G^i(x, y)$ are positively homogeneous of degree two in y^i and are given by

$$2G^i = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F), \qquad F = \frac{L^2}{2} \tag{2.1}$$

where g^{ij} are the inverse of g_{ij}. The well known Berwald connection $B\Gamma = (G^i_{jk}, G^i_j)$ of a Finsler space is constructed from the quantity G^i appearing in the equation of geodesic and is given by [8]

$$G^i_j = \dot{\partial}_j G^i, \qquad G^i_{jk} = \dot{\partial}_k G^i_j.$$

The Cartan's connection $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$ is constructed from the metric function L by the following five axioms [8]:

$$(i) \quad g_{ij|k} = 0 \quad (ii) \quad g_{ij}|_k = 0 \quad (iii) \quad F^i_{jk} = F^i_{kj} \quad (iv) \quad F^i_{0k} = G^i_k \quad (v) \quad C^i_{jk} = C^i_{kj},$$

where $_{|k}$ and $|_k$ denote $h-$ and $v-$covariant derivatives with respect to $C\Gamma$. It is clear that the $h-$covariant derivative of L with respect to $B\Gamma$ and $C\Gamma$ is the same and vanishes identically. Furthermore, the $h-$covariant derivatives of L_i, L_{ij} with respect to $C\Gamma$ are also zero.

We shall write

$$2r_{ij} = b_{i|j} + b_{j|i}, \qquad 2s_{ij} = b_{i|j} - b_{j|i}. \tag{2.2}$$

3 The $\beta-$change of Finsler Metric

The $\beta-$change of Finsler metric is given by

$$\overline{L}(x, y) = f(L, \beta), \tag{3.1}$$

where f is positively homogeneous function of degree one in L and β. Homogeneity of f gives

$$Lf_1 + \beta f_2 = f, \tag{3.2}$$

where subscripts '1' and '2' denote the partial derivatives with respect to L and β respectively.

Differentiating (3.2) with respect to L and β respectively, we get

$$Lf_{11} + \beta f_{12} = 0 \qquad \text{and} \qquad Lf_{12} + \beta f_{22} = 0.$$

Hence, we have

$$\frac{f_{11}}{\beta^2} = -\frac{f_{12}}{\beta L} = \frac{f_{22}}{L^2},$$

which gives

$$f_{11} = \beta^2 \omega, \qquad f_{22} = L^2 \omega, \qquad f_{12} = -\beta L \omega, \tag{3.3}$$

where Weierstrass function ω is positively homogeneous function of degree -3 in L and β. Therefore,

$$L\omega_1 + \beta\omega_2 + 3\omega = 0. \tag{3.4}$$

Again, ω_2 is positively homogeneous of degree - 4 in L and β, so

$$L\omega_{21} + \beta\omega_{22} + 4\omega_2 = 0.$$

Throughout the paper we frequently use the equations (3.2), (3.3) and (3.4) without quoting them. Also, we have assumed that f is not linear function of L and β so that $\omega \neq 0$.

We may put

$$\overline{G}^i = G^i + D^i. \tag{3.5}$$

Then, $\overline{G}^i_j = G^i_j + D^i_j$ and $\overline{G}^i_{jk} = G^i_{jk} + D^i_{jk}$, where $D^i_j = \dot{\partial}_j D^i$ and $D^i_{jk} = \dot{\partial}_k D^i_j$. The tensors D^i, D^i_j and D^i_{jk} are positively homogeneous in y^i of degree two, one and zero respectively. Therefore, we have

$$D^i_{jk} y^k = D^i_j, \qquad D^i_j y^j = 2D^i.$$

To find difference tensor D^i, we deal with equation [8] $L_{ij|k} = 0$, that means,

$$\partial_k L_{ij} - L_{ijr} G^r_k - L_{rj} F^r_{ik} - L_{ir} F^r_{jk} = 0. \tag{3.6}$$

Since $\dot{\partial}_i \beta = b_i$, from (3.1), we have

$$\begin{array}{lrcl}
(a) & \overline{L}_i & = & f_1 L_i + f_2 b_i, \\
(b) & \overline{L}_{ij} & = & f_1 L_{ij} + \beta^2 \omega L_i L_j - \beta L \omega (L_i b_j + L_j b_i) + L^2 \omega b_i b_j, \\
(c) & \partial_j \overline{L}_i & = & f_1 \partial_j L_i + (\beta^2 \omega L_i - \beta L \omega b_i) \partial_j L + (L^2 \omega b_i - \beta L \omega L_i) \partial_j \beta + f_2 \partial_j b_i, \\
(d) & \partial_k \overline{L}_{ij} & = & f_1 \partial_k L_{ij} + \{\beta^2 \omega L_{ij} + \beta^2 \omega_1 L_i L_j - (\beta L \omega_1 + \beta \omega)(L_i b_j + L_j b_i) + \\
 & & & (2L\omega + L^2 \omega_1) b_i b_j\} \partial_k L + \{-\beta L \omega L_{ij} + (2\beta\omega + \beta^2 \omega_2) L_i L_j - \\
 & & & (L\omega_1 + \beta L \omega_2)(L_i b_j + L_j b_i) + L^2 \omega_2 b_i b_j\} \partial_k \beta + \\
 & & & (\beta^2 \omega L_j - \beta L \omega b_j) \partial_k L_i + (\beta^2 \omega L_i - \beta L \omega b_i) \partial_k L_j - \\
 & & & (\beta L \omega L_j - L^2 \omega b_j) \partial_k b_i - (\beta L \omega L_i - L^2 \omega b_i) \partial_k b_j, \\
(e) & \overline{L}_{ijk} & = & f_1 L_{ijk} + \beta^2 \omega (L_i L_{jk} + L_j L_{ik} + L_k L_{ij}) - \beta L \omega (b_i L_{jk} + b_j L_{ik} + \\
 & & & b_k L_{ij}) + (\beta^2 \omega_2 + 2\beta\omega)(L_i L_j b_k + L_i L_k b_j + L_j L_k b_i) - \\
 & & & (\beta L \omega_2 + L\omega)(b_i b_j L_k + b_j b_k L_i + b_i b_k L_j) + \beta^2 \omega_1 L_i L_j L_k + L^2 \omega_2 b_i b_j b_k.
\end{array} \tag{3.7}$$

Since $\overline{L}_{ij|k} = 0$ in $\overline{F}^n$, after using (3.5), we have

$$\partial_k \overline{L}_{ij} - \overline{L}_{ijr}(G^r_k + D^r_k) - \overline{L}_{rj}(F^r_{ik} + {}^c D^r_{ik}) - \overline{L}_{ir}(F^r_{jk} + {}^c D^r_{jk}) = 0, \tag{3.8}$$

where $\overline{F}^i_{jk} - F^i_{jk} = {}^c D^i_{jk}$.

Substituting in the equation (3.8) the values of $\partial_k \overline{L}_{ij}$, $\overline{L}_{ir}$ and $\overline{L}_{ijr}$ from (3.7)(b),(d),(e) and using (3.6), we have

$$
\begin{aligned}
- \quad & f_1\{L_{ijr}D^r_k + L_{rj}\,D^r_{ik} + L_{ir}\,D^r_{jk}\} + \{\beta^2\omega L_{ij} + \beta^2\omega_1 L_iL_j - \\
& (\beta L\omega_1 + \beta\omega)(L_ib_j + L_jb_i) + (2L\omega + L^2\omega_1)b_ib_j\}L_rG^r_k + \\
& \{-\beta L\omega L_{ij} + (2\beta\omega + \beta^2\omega_2)L_iL_j - (L\omega + \beta L\omega_2) \times \\
& (L_ib_j + L_jb_i) + L^2\omega_2 b_ib_j\}(r_{0k} + s_{0k} + b_rG^r_k) + (\beta^2\omega L_j - L\beta\omega b_j) \times \\
& (L_{ir}G^r_k + L_rF^r_{ik}) + (\beta^2\omega L_i - L\beta\omega b_i)(L_{jr}G^r_k + L_rF^r_{jk}) - \\
& (L\beta\omega L_j - L^2\omega b_j)(r_{ik} + s_{ik} + b_rF^r_{ik}) - \\
& (L\beta\omega L_i - L^2\omega b_i)(r_{jk} + s_{jk} + b_rF^r_{jk}) - \\
& \{\beta^2\omega(L_iL_{jr} + L_jL_{ri} + L_rL_{ij}) - L\beta\omega(b_iL_{jr} + b_jL_{ir} + b_rL_{ij}) + \\
& (\beta^2\omega_2 + 2\beta\omega)(L_iL_jb_r + L_iL_rb_j + L_jL_rb_i) - \\
& (L\beta\omega_2 + L\omega)(L_rb_ib_j + L_ib_jb_r + L_jb_ib_r) + \\
& \beta^2\omega_1 L_iL_jL_r + L^2\omega_2 b_ib_jb_r\}(G^r_k + D^r_k) - \\
& \{\beta^2\omega L_rL_j - L\beta\omega(L_rb_j + L_jb_r) + L^2\omega b_rb_j\}(F^r_{ik} + {}^cD^r_{ik}) - \\
& \{\beta^2\omega L_rL_i - L\beta\omega(L_rb_i + L_ib_r) + L^2\omega b_rb_i\}(F^r_{jk} + {}^cD^r_{ik}) = 0,
\end{aligned}
\tag{3.9}
$$

where $\partial_k L = L_rG^r_k$, $\partial_k\beta = r_{0k} + s_{0k} + b_rG^r_k$, $\partial_k L_i = L_{ir}G^r_k + L_rF^r_{ik}$ and $\partial_k b_i = r_{ik} + s_{ik} + b_rF^r_{ik}$.

Contracting (3.9) with y^k, and using the fact that $D^i_{jk}y^j = {}^cD^i_{jk}y^j = D^i_k$ [2], we get

$$
\begin{aligned}
& 2\{f_1L_{ijr} + \beta^2\omega(L_iL_{jr} + L_jL_{ri} + L_rL_{ij}\} - \\
& L\beta\omega(b_iL_{jr} + b_jL_{ir} + b_rL_{ij}) + (\beta^2\omega_2 + 2\beta\omega)(L_iL_jb_r + \\
& L_iL_rb_j + L_jL_rb_i) - (L\beta\omega_2 + L\omega)(L_rb_ib_j + L_ib_jb_r + L_jb_ib_r) + \\
& \beta^2\omega_1 L_iL_jL_r + L^2\omega_2 b_ib_jb_r\}\,D^r + \{f_1L_{rj} + \beta^2\omega L_rL_j - \\
& L\beta\omega(L_rb_j + L_jb_r) + L^2\omega b_rb_j\}D^r_i + \{f_1L_{ir} + \beta^2\omega L_rL_i - \\
& L\beta\omega(L_rb_i + L_ib_r) + L^2\omega b_rb_i\}D^r_j + (L\beta\omega L_j - L^2\omega b_j) \times \\
& (r_{i0} + s_{i0}) + (L\beta\omega L_i - L^2\omega b_i)(r_{j0} + s_{j0}) + \{\beta L\omega L_{ij} - (2\beta\omega + \beta^2\omega_2)L_iL_j + \\
& (L\omega + L\beta\omega_2)(L_ib_j + L_jb_i) - L^2\omega_2 b_ib_j\}\,r_{00} = 0,
\end{aligned}
\tag{3.10}
$$

where $'0'$ stands for contraction with respect to y^i, viz. $r_{0k} = r_{ik}y^i$, $r_{00} = r_{ij}y^iy^j$.

Next, we deal with $\overline{L}_{i|j} = 0$, that is $\partial_j\overline{L}_i - \overline{L}_{ir}\overline{G}^r_j - \overline{L}_r\overline{F}^r_{ij} = 0$, then

$$
\partial_j\overline{L}_i - \overline{L}_{ir}(G^r_j + D^r_j) - \overline{L}_r(F^r_{ij} + {}^cD^r_{ij}) = 0. \tag{3.11}
$$

Putting the values of $\partial_j\overline{L}_i$, $\overline{L}_{ir}$ and $\overline{L}_r$ from (3.7) in (3.11) and using equation

$$
L_{i|j} = \partial_j L_i - L_{ir}G^r_j - L_rF^r_{ij} = 0,
$$

and rearranging the terms, we get

$$
\begin{aligned}
f_2 b_{i|j} = & \{fL_{ir} + \beta^2\omega L_iL_r + L^2\omega b_ib_r - L\beta\omega(L_ib_r + L_rb_i)\}\,D^r_j \\
& + (L\beta\omega L_i - L^2\omega b_i)(r_{0j} + s_{0j}) + (f_1L_r + f_2b_r)\,D^r_{ij},
\end{aligned}
$$

which after using (2.2) gives

$$
\begin{aligned}
2f_2r_{ij} \quad = \quad & \{f_1L_{ir} + \beta^2\omega L_iL_r + L^2\omega b_ib_r - L\beta\omega(L_ib_r + L_rb_i)\}\,D^r_j + \\
& \{f_1L_{jr} + \beta^2\omega L_jL_r + L^2\omega b_jb_r - L\beta\omega(L_jb_r + L_rb_j)\}\,D^r_i + \\
& (L\beta\omega L_i - L^2\omega b_i)(r_{0j} + s_{0j}) + (L\beta\omega L_j - L^2\omega b_j) \times \\
& (r_{0i} + s_{0i}) + 2(f_1L_r + f_2b_r)\,D^r_{ij}
\end{aligned}
\tag{3.12}
$$

and

$$\begin{aligned} 2f_2 s_{ij} &= \{f_1 L_{ir} + \beta^2 \omega L_i L_r + L^2 \omega b_i b_r - L\beta\omega(L_i b_r + L_r b_i)\}\, D^r_j - \\ &\quad \{f_1 L_{jr} + \beta^2 \omega L_j L_r + L^2 \omega b_j b_r - L\beta\omega(L_j b_r + L_r b_j)\}\, D^r_i + \\ &\quad (L\beta\omega L_i - L^2\omega b_i)(r_{0j} + s_{0j}) - (L\beta\omega L_j - L^2\omega b_j)(r_{0i} + s_{0i}). \end{aligned} \tag{3.13}$$

Subtracting (3.12) from (3.10) and contracting the resulting equation with y^i, we obtain

$$\begin{aligned} &\{-f_1 L_{jr} + L\beta\omega L_j b_r + L\beta\omega L_r b_j - \beta^2\omega L_j L_r - L^2\omega b_j b_r\}\, D^r - \\ &\frac{1}{2}(L\beta\omega L_j - L^2\omega b_j) r_{00} + f_2 r_{0j} = (f_1 L_r + f_2 b_r)\, D^r_j. \end{aligned} \tag{3.14}$$

Contracting (3.14) with y^j, we get

$$2(f_1 L_r + f_2 b_r)\, D^r = f_2 r_{00}. \tag{3.15}$$

Adding (3.10) and (3.13) and contracting the resulting equation with y^j, we get

$$\{f_1 L_{ir} + \beta^2\omega L_i L_r + L^2\omega b_i b_r - L\beta\omega(L_i b_r + L_r b_i)\}\, D^r \;=\; \frac{1}{2}(L^2\omega b_i - L\beta\omega L_i) r_{00} + f_2 s_{i0}. \tag{3.16}$$

In view of $LL_{ir} = g_{ir} - L_i L_r$, the equation (3.16) can be written as

$$\begin{aligned} &\frac{f_1}{L}\, g_{ir} D^r + \{\big(-\frac{f_1}{L} + \beta^2\omega\big) L_i - L\beta\omega b_i\}\, L_r D^r + \\ &(L^2\omega b_i - L\beta\omega L_i) b_r D^r = \frac{1}{2}(L^2\omega b_i - L\beta\omega L_i) r_{00} + f_2 s_{i0}. \end{aligned} \tag{3.17}$$

Contracting (3.17) with $b^i = g^{ij} b_j$, we get

$$\big(\frac{-f_1\beta}{L^2} - L\beta\omega\triangle\big)\, L_r D^r + \big(\frac{f_1}{L} + L^2\omega\triangle\big)\, b_r D^r = \frac{L^2\omega\triangle}{2} r_{00} + f_2 s_0, \tag{3.18}$$

where $\triangle = b^2 - \frac{\beta^2}{L^2}$ and $s_0 = s_{r0} b^r$.
The equations (3.15)) and (3.18) constitute the system of algebraic equations in $L_r D^r$ and $b_r D^r$ whose solution is given by

$$b_r D^r \;=\; \frac{(f_1 f_2 \beta + f L^3 \omega\triangle)}{2f(f_1 + L^3\omega\triangle)}\, r_{00} + \frac{f_1 f_2 L^2}{f(f_1 + L^3\omega\triangle}\, s_0 \tag{3.19}$$

and

$$L_r D^r \;=\; \frac{L f_1 f_2}{2f(f_1 + L^3\omega\triangle)}\, r_{00} - \frac{L^2 f_2^2}{f(f_1 + L^3\omega\triangle)}\, s_0. \tag{3.20}$$

Contracting (3.17) by g^{ij} and putting the values of $b_r D^r$ and $L_r D^r$ from (3.19) and (3.20) respectively, we get

$$\begin{aligned} D^i \;=\; &\left\{\frac{f_1 f_2 - f L\beta\omega}{2f(f_1 + L^3\omega\triangle)}\, r_{00} - \frac{L f_2 (f_1 f_2 - f L\beta\omega)}{f f_1 (f_1 + L^3\omega\triangle)}\, s_0\right\} y^i + \\ &\left\{\frac{L^3\omega}{2(f_1 + L^3\omega\triangle)}\, r_{00} - \frac{f_2 L^4 \omega}{f_1(f_1 + L^3\omega\triangle)}\, s_0\right\} b^i + \frac{L f_2}{f_1} s^i_0, \end{aligned} \tag{3.21}$$

where $l^i = \dfrac{y^i}{L}$.

Proposition 3.1. *The difference tensor $D^i = \overline{G}^i - G^i$ of any $\beta-$change of Finsler metric is given by (3.21).*

4 Projective Change of Finsler Metric

The Finsler space $\overline{F}^n$ is said to be projective to Finsler space F^n if every geodesic of F^n is transformed to a geodesic of $\overline{F}^n$. It is well known that the change $L \to \overline{L}$ is projective if $G^i = G^i + P(x,y)y^i$, where $P(x,y)$ is a homogeneous scalar function of degree one in y^i, called projective factor [3]. Thus, from (3.5) it follows that $L \to \overline{L}$ is projective iff $D^i = Py^i$.

Now, we consider that the $\beta-$change $L \to \overline{L} = f(L,\beta)$ is projective. Then, from equation (3.21), we have

$$\begin{aligned} Py^i &= \left\{ \frac{f_1 f_2 - fL\beta\omega}{2f(f_1 + L^3\omega\triangle)}\, r_{00} - \frac{Lf_2(f_1f_2 - fL\beta\omega)}{ff_1(f_1 + L^3\omega\triangle)}\, s_0 \right\} y^i + \\ & \left\{ \frac{L^3\omega}{2(f_1 + L^3\omega\triangle)}\, r_{00} - \frac{f_2 L^4\omega}{f_1(f_1 + L^3\omega\triangle)}\, s_0 \right\} b^i + \frac{Lf_2}{f_1} s^i_0, \end{aligned} \tag{4.1}$$

Contracting (4.1) with $y_i(= g_{ij}y^j)$ and using the fact that $s^i_0 y_i = 0$ and $y_i y^i = L^2$, we get

$$P = \frac{f_1 f_2}{2f(f_1 + L^3\omega\triangle)}\, r_{00} - \frac{f_2^2 L}{f(f_1 + L^3\omega\triangle)}\, s_0. \tag{4.2}$$

Putting the value of P from (4.2) in (4.1), we get

$$\beta\omega(f_1 r_{00} - 2f_2 L s_0)\, y^i = L^2\omega(f_1 r_{00} - 2f_2 L s_0)\, b^i + 2f_2(f_1 + L^3\omega\triangle)\, s^i_0. \tag{4.3}$$

Transvecting (4.3) by b_i, we get

$$r_{00} = \frac{-2f_2}{L^2\omega}\frac{s_0}{\triangle}, \qquad \text{where} \qquad \triangle = b^2 - \frac{\beta^2}{L^2} \neq 0. \tag{4.4}$$

Substituting the value of r_{00} from (4.4) in (4.2), we get

$$P = \frac{-f_2^2}{fL^2\omega}\frac{s_0}{\triangle}. \tag{4.5}$$

Eliminating P and r_{00} from (4.5), (4.4) and (4.1), we get

$$s^i_0 = \left(b^i - \frac{\beta}{L^2}y^i\right)\frac{s_0}{\triangle}. \tag{4.6}$$

The equations (4.4) and (4.6) give the necessary conditions under which the $\beta-$change becomes a projective change. Conversely, if conditions (4.4) and (4.6) are satisfied, then putting these conditions in (3.21), we get

$$D^i = \frac{-f_2^2}{fL^3\omega}\frac{s_0}{\triangle}y^i, \quad \text{i.e.} \quad D^i = Py^i, \quad \text{where} \quad P = \frac{-f_2^2}{fL^3\omega}\frac{s_0}{\triangle}.$$

Thus $\overline{F}^n$ is projective to F^n.

Theorem 4.1. *The $\beta-$change of Finsler space is projective if and only if (4.4) and (4.6) hold.*

Let us assume that L is the metric of a Riemannian space, that is, $L = \alpha = \sqrt{a_{ij}(x)y^iy^j}$. Then $\overline{L} = f(\alpha,\beta)$ which is the metric of any $\beta-$changed space. In this case $b_{i|j} = b_{i;j}$ where $;j$ denotes the covariant derivative with respect to Christoffel symbols constructed from Riemannian metric α. Thus r_{ij} and s_{ij} are functions of coordinates only, and in view of the Theorem 4.1, it follows that the Riemannian space is projective to Finsler space obtained from $\beta-$change (3.1) iff

$$r_{00} = \frac{-2f_2}{\alpha^2\omega}\frac{s_0}{\triangle} \qquad \text{and} \qquad s^i_0 = -\big(\frac{\beta}{\alpha^2}y^i - b^i\big)\,\frac{s_0}{\triangle}, \quad \text{where} \quad \triangle = b^2 - \frac{\beta^2}{\alpha^2} \neq 0.$$

These equations may be written as

$$\text{(a) } r_{00}\,(\beta^2 - b^2\alpha^2) = \frac{2f_2}{\omega} s_0, \qquad \text{(b)} \quad s_0^i\,(\beta^2 - b^2\alpha^2) = (\beta y^i - \alpha^2 b^i)s_0. \tag{4.7}$$

The equation (4.7)(b) can be written as

$$\begin{aligned}(s_j^i b_h b_k + s_h^i b_j b_k + s_k^i b_j b_h) - b^2(s_j^i a_{hk} + s_h^i a_{jk} + s_k^i a_{jh}) \\ = \frac{1}{2}[(b_h s_k + b_k s_h)\delta_j^i + (b_j s_k + b_k s_j)\delta_h^i + \\ (b_j s_h + b_h s_j)\delta_k^i] - b^i(a_{hk}s_j + a_{hj}s_k + a_{jk}s_h).\end{aligned}$$

Contracting this equation with $i = j$, we get

$$(s_h b_k + s_k b_h) = 0, \quad \text{for} \;\; n > 2. \tag{4.8}$$

Transvection of (4.8) by b^h, we get $b^2 s_k = 0$, which implies that $s_k = 0$ provided $b^2 \neq 0$. Therefore, we have $s_0^i = 0$, $s_0 = 0$ and (4.7)(a) gives $r_{00} = 0$ as $\beta^2 - b^2\alpha^2 \neq 0$, consequently $r_{ij} = 0$, $s_{ij} = 0$. Hence $b_{i;j} = 0$, i.e. the pair (α, β) is parallel pair. Conversely, if $b_{i;j} = 0$ the equation (4.7)(a) and (b) hold identically. Thus, we have the following theorem

Theorem 4.2. *The Riemannian space with metric α is projective to the Finsler space with $(\alpha, \beta)-$metric iff the (α, β) is a parallel pair.*

5 Particular Cases of One-form β

Let b_i be components of a parallel vector field in F^n i.e. $b_{i|j} = 0$. Therefore $r_{ij} = s_{ij} = 0$. Hence, the equation (3.21) gives $D^i = 0$ which implies that $\overline{G}^i = G^i$, $\overline{G}_j^i = G_j^i$ and $\overline{G}_{jk}^i = G_{jk}^i$. Thus, we have the following theorem

Theorem 5.1. *The Berwald connection $B\Gamma = (G_{jk}^i, G_j^i, 0)$ is invariant under $\beta-$change for parallel vector field b_i.*

Let b_i be a concurrent vector field in F^n [4, 9]. Then, we have (i) $\;b_{i|j} = -g_{ij}$ $\qquad$ (ii) $\;b_i C_{jk}^i = 0$.

Thus, for a concurrent vector field $s_{ij} = 0$, $r_{ij} = -g_{ij}$ and therefore $r_{00} = -L^2$ and equation (3.21) reduces to

$$D^i = -\frac{(f_1 f_2 - f L\beta\omega)L^2}{2f(f_1 + L^3\omega\triangle)}\, y^i - \frac{L^5\omega}{2(f_1 + L^3\omega\triangle)}\, b^i. \tag{5.1}$$

If $D^i = 0$, then equation (5.1) shows that b^i and y^i are linearly related. That is, there exists a scalar λ such that

$$b^i = \lambda y^i. \tag{5.2}$$

Since for a concurrent vector field the contravariant components b^i are also functions of x^i only [4, 9], differentiating (5.2) with respect to y^j, we get

$$(\dot{\partial}_j \lambda) y^i + \lambda \delta_j^i = 0.$$

Contracting it with respect to i and j and using the fact that λ is homogeneous function of degree -1 in y^i, we get

$$(n-1)\lambda = 0 \quad \text{i.e.} \quad \lambda = 0$$

which is not possible as in that case β vanishes. Hence, we have the following theorem

Theorem 5.2. *If b_i are components of a concurrent vector field, then the Berwald connection $B\Gamma$ is not invariant under $\beta-$change.*

Next, suppose that b_i is a gradient vector field so that $s_{ij} = \frac{1}{2}(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}) = 0$. Then equation (3.21) reduces to

$$D^i = \frac{(f_1 f_2 - fL\beta\omega)}{2f(f_1 + L^3\omega\triangle)} r_{00} y^i + \frac{L^3\omega r_{00}}{2(f_1 + L^3\omega\triangle)} b^i. \tag{5.3}$$

If $r_{00} \neq 0$ and $D^i = 0$, then we get the same result as given in theorem (5.2) for gradient vector field b_i but if r_{00} is also zero, then the Berwald connections $B\Gamma$ is invariant under $\beta-$change. Thus, we have the following theorem

Theorem 5.3. *If b_i are components of a gradient vector field, then the Berwald connection $B\Gamma$ is invariant if $r_{00} = 0$.*

6 Conclusion

Here, I introduced n-dimension Finsler space differentiable on a manifold along with a fundamental function. With some preliminaries and historical developments, we mainly focused on the β-change of Finsher metric and projective change of Finsler metric. The difference tensor of any β-change of Finsher metric is derived. The necessary and sufficient condition of the β-change of a Finsher space to be projective is presented. Some particular cases of β-change and Berwald connections are discussed.

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References

[1] Matsumoto M (1974) On Finsler space with Randers metric and special forms of important tensors, *J. Math. Kyoto Univ.*, **14**, 477-498.

[2] Matsumoto M (1986) *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Otsu, Japan.

[3] Matsumoto, M (1992) Theory of Finsler spaces with $(\alpha, \beta)-$ metric, *Rep. Math. Phy.*, **31**, 43-83.

[4] Matsumoto M and Eguchi K (1974) Finsler space admitting a concurrent vector field, *Tensor, N. S.*, **28**, 239-249.

[5] Park HS and Lee IY (2001) The Randers changes of Finsler spaces with $(\alpha, \beta)-$ metrics of Douglas type, *J. Korean Math. Soc.*, **38**, 503-521.

[6] Prasad, B. N. and Kumari, B., 2013, The $\beta-$change of Finsler metric and imbedding classes of their tangent spaces, *Tensor, N. S.*, **74**(1): 48-59.

[7] Prasad BN, Shukla HS and Pandey OP (2013) The exponential change of Finsler metric and relation between imbedding class numbers of their tangent Riemannian spaces, *Romanian Journal of Mathematics and Computer Science*, **3**(1):96-108.

[8] Shukla HS, Pandey OP and Joshi HD (2012) Matsumoto change of Finsler metric, *J. Int. Acad. Phys. Sci.*, **16**(4):329-341.

[9] Tachibana S (1950) On Finsler spaces which admit a concurrent vector field, *Tensor, N. S.*, **1**:1-5.