Generalized Common Fixed Point Result in Banach Space

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Abstract: *The aim of this paper is to establish a common fixed point theorem for weakly compatible pair of self mappings in Banach space which generalizes and improves various well known comparable results.*

Keywords: Fixed point, Banach space, compatible mappings.

1 Introduction

The study of common fixed points of mappings satisfying certain contractive type conditions has been a very active field of research. In 1982, Sessa [12] defined weak commutativity and proved a common fixed point theorem for weakly commuting mappings. Also, Jungck [4] introduced more generalized commutativity, so-called compatibility, which is more general than that of weak commutativity. Further, in 1998, Jungck and Rhoades [5] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but the converse need not be true.

Some theorems on unique fixed points for expansive mappings are proved by Popa [8]. Popa [9] further extended results [8] for compatible mappings. In 1999, Popa [10] proved some common fixed point theorems for compatible mappings satisfying an implicit relation. For more common fixed point results in metric space using implicit relation, one can refer Aliouche et al. [1], Imdad et al. [3], Kumar and Jha [6], Pathak et al. [7] and Popa [11].

The main purpose of this paper is to prove a common fixed point theorem for weakly compatible mappings in Banach Space satisfying an implicit relation without employing continuity condition. Our result generalizes and improves various other similar results of fixed points. We also give an example to illustrate our result.

Definition 1.1 (Weak Compatibility of self-mappings)
Two self mappings A and S of a metric space (X, d) are said to be weakly compatible if they commute at coincident points. That is, Ax = Sx implies that ASx = SAx for all x in X. **Lemma 1.1**[4] Let S and T be weakly compatible self mappings on a metric space (X, d). If St = Tt for some $t \in X$ then we have

$$STt = TSt.$$

We start with the following implicit relations:
Let Φ be the set of all real continuous functions $\phi(t_1, t_2, \cdots, t_6) = \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:
$\phi_1 : \phi$ is non increasing in variable t_6.
ϕ_2: there exists $h \in (0, 1)$ such that for every $u, v \geq 0$ with
$\phi_a : \phi(u, v, v, u, (u + v)/2, 0) \leq 0$
or
$\phi_b : \phi(u, v, u, v, (u + v)/2, u + v) \geq 0$, we have $u \leq hv$.
$\phi_3 : \phi(u, u, 0, 0, 0, u) > 0$ for all $u > 0$.

Example 1.1
$\phi(t_1, t_2, \cdots, t_6) = t_1 - kmax\{t_2, t_3, t_4, t_5, (1/2)t_6\}$ where $k \in (0, 1)$.
ϕ_1 : obvious.
ϕ_2 : Let $u > 0$, $\phi(u, v, v, u, (u + v)/2, 0) = u - kmax\{v, v, u, (u + v)/2, 0\} \leq 0$. If $u \geq v$, then $u \leq ku < u, a$ contadiction.
Thus, we have $u < v$ and $u \leq ku = hv$ where $h = k \in (0, 1)$
Similarly, if $u > 0$ then $\phi(u, v, v, u, (u+v)/2, 0) \leq 0$ which implies that $u \leq hv$. Also, if $u = 0$, then $u \leq hv$.

$\phi_3 : \phi(u, u, 0, 0, 0, u) = (1 - k)u > 0$ for all $u > 0$.
The following examples can be used the support above lemma.

Example 1.2
$\phi(t_1, t_2, \cdots, t_6) = t_1^2 - \alpha\{t_2^2 - t_6((1/2)(t_3 + t_4) - t_5)\}$ where $\alpha \in (0, 1)$ where $k \in (0, 1)$.
ϕ_1 : obvious.
ϕ_2 : Let $u > 0$, $\phi(u, v, v, u, (u+v)/2, 0) = u^2 - av^2 \leq 0$ which implies that $u \leq v\sqrt{a} = hv$ where $h = \sqrt{a} < 1$.
Similarly, if $u > 0$ then $\phi(u, v, v, u, (u+v)/2, 0) \leq 0$ which implies that $u \leq hv$. Also, if $u = 0$, then $u \leq hv$.
$\phi_3 : \phi(u, u, 0, 0, 0, u) = u^2(1 - a) > 0$ for all $u > 0$.

Example 1.3
$\phi(t_1, t_2, \cdots, t_6) = t_1^2 - c_1 max\{t_2^2, t_3^2, t_4^2\}$
$- c_2 max\{t_3t_5, (1/2)(t_4t_6)\}$ where $c_1 > 0, c_2 > 0, c_1 + c_2 < 1$.
ϕ_1 : obvious.
ϕ_2 : Let $u > 0$, $\phi(u, v, v, u, (u + v)/2, 0) = u^2 - c_1 max\{v^2, v^2, u^2\}$
$- c_2 max\{v(1/2)(u + v), 0\} \leq 0$.
If $u \geq v$ then $u^2(1 - c_1 - c_2) \leq 0$ which implies that $c_1 + c_2 \geq 1$ a contradiction. Thus we have $u < v$ and $u \leq v\sqrt{(c_1 + c_2)} = hv$ where $h = \sqrt{(c_1 + c_2)} < 1$.
Similarly, if $u > 0$ then $\phi(u, v, v, u, (u + v)/2, 0)$ implies that $u \leq hv$.
$\phi_3 : \phi(u, u, 0, 0, 0, u) = u^2(1 - c_1) > 0$ for all $u > 0$.

2 Main results

Theorem 2.1
Let$(X, ||.||)$be a Banach space and $A, B, S, T : X \to X$ be mappings satisfying the conditions:
(i) $AX \subset TX$ and $BX \subset SX$, and
(ii) $\phi(||Ax - By||, ||Sx - Ty||, ||Ax - Sx||, ||By - Ty||, (||Ax - Sx|| + ||By - Ty||)/2,$
$||Ax - Ty||) \leq 0$, for all x, y in X where $\phi \in \Phi$.
If one of SX, TX, AX or BX is a complete subspace of X and {A, S} and {B, T} are weakly compatible pairs, then A, B, S and T have a unique common fixed point.

Proof Since $AX \subset TX$, using condition (i) for an arbitrary point x_0 in X there exists a point x_1 in X such that $Ax_0 = Tx_1$. Also, since $BX \subset SX$, for this point x_1 in X we can choose a point x_2 in X such that $Bx_1 = Sx_2$, and so on. Inductively, we can define a sequences$\{x_n\}$ and $\{y_n\}$ in X such that
$y_{2n} = Tx_{2n+1} = Ax_{2n} and y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ for every $n = 0, 1, 2, ...$
Applying relation (ii), we have
$\phi(||Ax_{2n} - Bx_{2n+1}||, ||Sx_{2n} - Tx_{2n+1}||, ||Ax_{2n} - Sx_{2n}||, ||Bx_{2n+1} - Tx_{2n+1}||,$
$(||Ax_{2n} - Sx_{2n}|| + ||Bx_{2n+1} - Tx_{2n+1}||)/2, ||Ax_{2n} - Tx_{2n+1}||) \leq 0,$
this implies
$\phi(||Ax_{2n} - Bx_{2n+1}||, ||Ax_{2n} - Bx_{2n-1}||, ||Ax_{2n} - Bx_{2n-1}||, ||Ax_{2n} - Bx_{2n+1}||,$
$(||Ax_{2n} - Bx_{2n-1}|| + ||Ax_{2n} - Bx_{2n+1}||)/2, 0) \leq 0.$
By (ϕ_a), we have$||Ax_{2n} - Bx_{2n+1}|| \leq h||Ax_{2n} - Bx_{2n-1}||$.
Similarly, applying(ϕ_1) and (ϕ_b), we get $||Ax_{2n} - Bx_{2n-1}|| \leq h\ ||Ax_{2n-2} - Bx_{2n-1}||$and so for $n = 0, 1, 2, 3, ...$ we have $||Ax_{2n} - Bx_{2n-1}|| \leq h^{2n}||Ax_0 - Bx_1||$.
It follows that y_n is a Cauchy sequence in X.
Now, suppose SX is complete. The sequence$\{y_{2n+1}\}$ is contained in SX and has a limit , say u, in SX. Let $v \in S^{-1}u$. Then, we get Sv = u. Also, the subsequence $\{y_{2n}\}$ also converges to u. Now, we claim that Av = u. If $Av \neq u$, then using relation (ii), we get
$\phi(||Av - Bx_{2n+1}||, ||Sv - Tx_{2n+1}||, ||Av - Sv||, ||Bx_{2n+1} - Tx_{2n+1}||,$
$(||Av - Sv|| + ||Bx_{2n+1} - Tx_{2n+1}||)/2, ||Av - Tx_{2n+1}||) \leq 0.$
As $n \to \infty$, by the continuity of ϕ, we get
$\phi(||Av - u||, 0, ||Av - u||, 0, (||Av - u|| + 0)/2, ||Av - u||) \leq 0$
which implies by (ϕ_b) that Av = u. This means that u is in the range of A and since $AX \subset TX$, there exists w in X such that Tw = u.
Now, we show that Bw = u. If$Bw \neq u$, then by using (ii), we have

$\phi(||Av-u||,||Sv-Tw||,||Av-Sv||,||Bw-Tw||,(||Av-Sv||+||Bw-w||)/2,$
$||Av-Tw||)\leq 0.$
As $n\to\infty$, by the continuity of ϕ, we get
$\phi(0,0,||u-Bw||,0,(||u-Bw||+0)/2,0)\leq 0)$
which implies by (ϕ_b) that Bw = u. Thus, we have Av = Sv = u = Tw = Bw. Since {A, S} and {B, T} are weakly compatible at v and w respectively, so we get
Au = ASv = SAv = Su and Bu = BTw = TBw = Tu.
Thus, from relation (ii), we have
$\phi(||Au-Bu||,||Su-Tu||,||Au-Su||,||Bu-Tu||,(||Au-Su||+||Bu-Tu||)/2,$
$||Au-Tu||)\leq 0.$
This implies $\phi(||u-Tu||,||u-Tu||,0,0,0,||u-Tu||)\leq 0$,
which contradicts to (ϕ_3) if $||u-Tu||\neq 0$. Thus, we have Tu = u.
Similarly, we can prove that Au = u. Hence, we get u = Au = Su = Tu = Bu.
We have therefore proved that u is a common fixed point of A, B, S and T. Again, the uniqueness of the common fixed point follows easily using condition (ii).
This completes the proof of the theorem.
Now, we give the following example to support above result.

Example 2.1
Let X = [2, 20] with the usual norm. Define A, B, S and T by
$A=B:X\to X$ by $Bx=x$ if $x=2$ or $>5, Bx=6$ if $2<x\leq 5$;
and $S=T:X\to X$ by $Tx=x$ if $x=2, Tx=12$ if $2<x\leq 5, Tx=x-3$ if $x>5$.
Then, the pairs (A, S) and (B, T) are weakly compatible mappings. Also, the mappings satisfy all the conditions of above theorem and they have a unique common fixed point x = 2.

3 Conclusion

As the fixed point result has been established with minimal contractive condition, so it improves and generalizes the results of Popa [7, 9, 10] and Sharma and Deshpandey [13]. It also extends the result of Chugh and Kumar [2] and other similar results for fixed point in Banach space.

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