On Fermat-Torricelli Problem in Frechet Spaces

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Abstract: *We study the Fermat-Torricelli problem (FTP) for Frechet space X, where X is considered as an inverse limit of projective system of Banach spaces. The FTP is defined by using fixed countable collection of continuous seminorms that defines the topology of X as gauges. For a finite set A in X consisting of n distinct and fixed points, the set of minimizers for the sum of distances from the points in A to a variable point is considered. In particular, for the case of collinear points in X, we prove the existence of the set of minimizers for FTP in X and for the case of non collinear points, existence and uniqueness of the set of minimizers are shown for reflexive space X as a result of strict convexity of the space.*

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1 Introduction

In 1643, Pierre de Fermat [17], proposed a problem to an Italian Physicist, Evangelista Torricelli. The problem was about finding the point the sum of whose distances from the vertices of a triangle is a minimum. The minimum value is known as Fermat-Torricelli value or Torricellian point. (See Mordukhovich and Nam [10]). The study of FTP together with its generalization has attracted several researchers due to its applications in location science and optimal networks.

The following authors worked on the FTP in normed linear spaces (nls), reflexive nls, reflexive Banach spaces, inner product spaces (i.p.s) and normed planes and spaces respectively: Radulescu et al. [14], Vesely [21], Papini and Puerto [13], Dragomir and Comanescu [3], Dragomir et al. [4] and Matrini et al. [9]. In recent time, Radulescu et al. [14] obtained several general formulations of FTP in nls by using theory of convex analysis and optimization. Vesely [21] established results on the FTP in nls that are reflexive. Papini and Puerto [13] studied the problem in Banach spaces by minimizing the sum of distance from k furthest point of the domain. Dragomir and Comanescu [3] presented results in i.p.s. An interesting version for solving the problem in reflexive nls, i.p.s and non-expansive spaces is proposed in Dragomir et al. [4]. However, constrained version of the problem using the distance penalty method can be found in Nathan and George [11]. As a corollary, the study also dealt with further generalization of the FTP. Meanwhile, a re visitation of FTP was done from the theoretical and numerical point of view by Boris and Nguyen [2] very recently.

In this paper, to further expand the scope of domain of definition of the functional, we aim to formulate FTP in more generalized spaces, namely Frechet spaces. In particular, we consider the existence of Torricellian points in Frechet spaces and as a proposition, we state the properties of functional which recaptures similar results in Dragomir and Comanescu ([3], proposition 1). This work complements the results of authors in [3, 4, 14] and also extends Ayinde and Osinuga [1]. The main aim of this paper is to investigate and generalize Proposition (2.3) of [13], Propositions (4.2, 4.3 and 5.1) of [4] for the case when X is a Frechet space.

In section 2, we give notions, definitions and results that have direct bearing to our work. In sections 3, we recall general results of FTP in normed linear spaces that we intend to generalize for Frechet spaces. Some preliminary results that lead to the existence results on FTP in Frechet spaces are considered first in subsection 4.1. These preliminary results include Theorem 4.1, which shows the existence of FTP in Frechet space vis-a-vis its existence in Banach space, Proposition 4.1 for the construction of bounded set for FTP in Frechet space and Proposition 4.2 which establishes the properties of functional described by the FTP.

Existence results in the case where the points are collinear are examined in subsection 4.2. Subsection 4.3 is concerned with the main results on the existence of the set of minimizers in Frechet spaces in the case of non-collinear points. Proposition 4.5 addresses the properties of the set of minimizers for the FTP, Theorem 4.2 and Proposition 4.6 are given to show the roles weak compactness and weak lower semi-continuity play on the existence of the set of minimizers for FTP while in Theorem 4.3 the existence result is established. In subsection 4.4 uniqueness of the minimizer for non collinear points in Frechet spaces is established. Strict convexity concept of the space and the functional FTP is used to show the uniqueness of the solution. This can be seen in Theorems 4.5 and 4.6.

2 Preliminaries

We shall give here a brief review of terminologies and results about linear topological spaces and Frechet spaces required for subsequent developments. The details can be found in ([5-8], [12], [15], [16], [18-20] and [22]).

Definition 2.1. *A family $\{p_i\}_{i\in I}$ of continuous seminorms on X is called a fundamental system of seminorms, if the sets $U_i = \{x \in X : p_i(x) < \epsilon\}(i \in I)$ form a fundamental system of zero neighborhoods.*

Definition 2.2. *A locally convex space(lcs) is a topological linear space with fundamental system of 0-neighbourhoods comprising convex sets. Every lcs X has a fundamental system of continuous seminorms $\{p_i\}_{i\in I}$*

Definition 2.3. *A lcs X is called a metrizable locally convex space if its topology is defined by a countable collection of continuous semi norms. If X is complete we call it a Frechet space.*

Definition 2.4. *A topological linear space X is referred to as a normed linear space (nls) if its topology is defined by a norm $\|\cdot\|$. A normed linear spce X which is complete is called a Banach space.*

Definition 2.5. *Let X be a vector space over K and X_i family of lcs. Then X together with the collection X_i and linear maps $f_i : X \to X_i,\ i \in \wedge$ is called a projective system if for each $s \in X,\ s \neq 0$, there exists an $i \in \wedge$ with $f_i(s) \neq 0$.*

For every projective system $(f_i : X \to X_i)$ $i \in \wedge$, the seminorm system $\{p : p = \max_i(p_i \circ f_i), i \in \wedge \quad p_i$ continuous seminorm on $X_i,\ i \in \wedge\}$ induced a locally convex topology on X, which is denoted by the projective limit topology of the system.

Definition 2.6. *A pair $(X, (f_i),\ i \in \wedge)$ consisting of lcs X and a family of $f_i \in L(X, X_i)$ is called the projective limit of a projective system $\{X_i,\ i \in \wedge, f_{ij},\ i > j\}$ which is denoted by $X = \lim_{\leftarrow} X_i$ where $f_{ij} : X_i \to X_j$ is continuous if the family (f_i) is compatible with (f_{ij}), i.e. $f_{ij} \circ f_i = f_j$.*

Definition 2.7. *A function $h : X \to (0, \infty)$ is called convex if for any $s, t \in X$ and $\lambda \in [0, 1]$, we have $h(\lambda s + (1 - \lambda)t) \leq \lambda h(s) + (1 - \lambda)h(t)$. If the inequality above is $<$ whenever $s \neq t$ and $\lambda \in (0, 1)$, then h is called strictly convex.*

Definition 2.8. *Let X be an nls. X is strictly convex iff $\|s + t\| = \|s\| + \|t\|$, $s \neq t$, then $t = \lambda s$ for some $\lambda > 0$, $s, t \in X$. (See [2]) (or X is strictly convex if the norm is strictly convex.)*

Definition 2.9 (Lower semi continuity). *If $f : X \to [-\infty, \infty]$ is an extended real valued function, we say f is lower semi continuous at s_0 if $f(s_0) \leq \lim_{s\to s_0} \inf f(s)$.*

The Frechet space X considered in this work is graded. This implies that X is equipped with a fixed fundamental system of seminorms whereby its topology is given by an increasing sequence of norms $p_n = ||.||_n$, $n > 0$ for all $x \in X$, $||x||_n \leq ||x||_{n+1}$
where $X_n = X/kerp_n$ is a Banach space. Hence, this suggests the following projective system.

$$\to X_{n+1} \to X_n \to X_{n-1} \to ... \to X_0$$

Each identity map is continuous and injective.

We remark that $X = \lim_{\leftarrow} X_n = \bigcap X_n$ is a dense subspace of X_n
Therefore, $U_n = \{x \in X : p_n(x) \leq \alpha\}$ $(n \in \mathbb{N})$ is a fixed fundamental system of neighbourhoods with fixed fundamental system $\{p_n\}_{n\in\mathbb{N}} = \mathcal{P}$ of seminorms/norms.

We let $X_n = X/kerp_n$ be a Banach space which is of finite dimension.

Let X be a graded Frechet space with the Banach space $X_n = X/kerp_n$ finite dimensional. Given a convex and closed set $U_n \subset X$ with $0 \in U_n$, the gauge function of the set U_n

$p_{U_n} : X \to \mathbb{R}^+$ is defined as

$$p_{U_n}(x) = \begin{cases} +\infty \quad if \quad \{\alpha > 0 : x \in \alpha U_n\} = \phi \\ \\ inf\{\alpha > 0 : \quad x \in \alpha U_n\} \quad otherwise \end{cases}$$

The gauge function $p_{U_n} = p_n$ is a seminorm. This implies that if $0 \in U_n$, domain of p_n is X.

Definition 2.10. *Let X be a locally convex space. A subset $U \subset X$, is called bornivorous if for every bounded set $B \subset X$ there is $\lambda > 0$ so that $B \subset \lambda U$. X is called bornological if every absolutely convex bornivorous set is a neighborhood of zero.*

3 General Results

We recall in this section, results on the existence and uniqueness of the set of minimizers for FTP in normed linear spaces and inner product spaces.

3.1 FTP in normed linear spaces

The following results on the existence and uniqueness of the set of minimizers for FTP in normed linear spaces are stated here. The proofs can be found in [4] and [11].

Definition 3.1. *Let $(X, ||.||)$ be a real normed linear space, let $m \geq 1$ be a positive integer and let $A = \{a_1, ..., a_m\}$ be a set of m distinct points in X, then the FTP associated with A on X is $T(s) = \sum_{i=1}^{m} ||(x_0 - a_i)||$ and its solution set is given by $T_X = \{x_0 : T(x_0) \leq T(x), \quad x \in X\}$.*

Proposition 3.1 (*cf* Papini and Puerto [11 Proposition 2.3])**.** *Let X be a nls. If X is a dual space, in particular, if X is reflexive, then for all $\{a_1, ..., a_n\}$ a set of distinct points in X, the set of minimizers for FTP is non-empty.*

Proposition 3.2 (*cf* Dragomir et al. [4 Proposition 5.1])**.** *Let $(X, ||.||$ be a nls and let $A = \{a_1, ..., a_n\} \subset X$ $(n \geq 3)$ be a set of non-collinear points in X. If the space is strictly convex, then the set of minimizers for FTP contains at most one point.*

Proposition 3.3 (*cf* Dragomir et al. [4 Proposition 5.1])**.** *Let $(X, ||.||]$ be a nls and let $A = \{a_1, ..., a_n\} \subset X$ $(n \geq 3)$ be a set of non-collinear points in X. If the space is strictly convex, then the set of minimizers for FTP contains at most one point.*

Proposition 3.4 (*cf* Dragomir et al. [4 Proposition 4.3])**.** *Let $(X, ||.||)$ be a nls and let $A = \{a_1, ..., a_{2k}\}$ be a set of $2k$ collinear distinct points in X. Then, the set of minimizers for the FTP is $\{a_{k+1}\}$.*

3.2 FTP in inner product spaces

We record here results on the existence and uniqueness of the set of minimizers for the FTP in inner product spaces. We also note that it suffices to state the following theorem which covers for both the existence and uniqueness of the set of minimizers since an inner product space is already a strictly convex space.

Proposition 3.5 (cf Dragomir et al. [4 Proposition 5.2])**.** *Let $(X(.,.))$ be an inner product space and let $A = \{a_1, ..., a_n\} \subset X$ $(n \geq 3)$ be a set of non-collinear points in X. Then the set of minimizers for FTP contains a unique point.*

4 Main Results

In this section, we generalise existence results of Propositions (3.3 and 3.4) for collinear points that are available in normed linear spaces to Frechet spaces and also existence and uniquiness results of propositions (3.1 and 3.2) for non collinear points available in normed linear spaces to Frechet spaces.

4.1 Preliminary results

We consider here the general results on the properties of the FTP functional.

Definition 4.1. *Let X be a real Frechet space. Let $m \geq 1$ and $A = \{q_1, q_2, \dots, q_m\} \subset X$ be a finite distinct set. If $\mathcal{P} = \{p_n\}_{n\in\mathbb{N}}$ are the fixed gauges of X, then, the FTP is given by $N(s) = \sum_{i=1}^{m} p_n(s - q_i)$ where $p_n \in \mathcal{P}$ for all n.*

Theorem 4.1. *Let X be a Frechet space (projective limit of Banach spaces X_i). Then the following conditions are equivalent.*

(i) There is a FTP defined on X;

(ii) For each Banach space X_i with continuous map $g_i : X \to X_i$ there exists a FTP defined on each Banach space X_i.

Proof. (i) implies (ii):

Define for each Banach space X_i a ball $B_i \subset X_i$. Let also find balls $B_1 \subset B_i$ and $B_2 \subset B_1$ such that $B_1 + B_1 \subset B_i$. Let U be an absolutely convex 0-neighbourhood in X such that

$$U = g_i^{-1}(B_2)$$

.

Let $r > 0$ and also w.l.o.g. $r \geq 1$. Given a finite set $M' = \{a_1, a_2, \dots, a_m\} \subset X_i$ we define a finite set $M = \{q_1, q_2, \dots, q_m\} \subset X$ with

$$M' \subset g_i(M) + r^{-1}(B_2).$$

Then there is

$$s \in rU \text{ where } s - q_j \in U \text{ for each } q_j \in M$$

whereby

$$\sum_{j=1}^{m} p_i(s - q_j)$$

where $p_i \in \mathcal{P}$ for all i exists in X by assumption.
Hence, from

$$U = g_i^{-1}(B_2), \text{ we have } g_i(s - q_j) \in B_2 \text{ for } q_j \in M$$

which exactly means that for each i,

$$||g_i(s - q_j)||_i < 1 \text{ and } \textstyle\sum_{j=1}^{m} ||g_i(s - q_j)||_i \text{ in } X_i.$$

Taking any $a_j \in M'$ and choose $q_j \in M$ which satisfies

$$a_j - g_i(q_j) \in r^{-1}B_2$$

that is

$$||a_j - g_i(q_j)||_i < r^{-1}.$$

Therefore,

$$s' = g_i(s) \in rB_2 \subset rB_i$$

and this means

$$||s'||_i = ||g_i(s)||_i < r,$$

for each i. Therefore,

$$||s' - a_j||_i = ||g_i(s - q_j)||_i + ||g_i(q_j) - a_j||_i < 1 + r^{-1},$$

for each i which finally gives

$$\sum_{j=1}^{m} ||s' - a_j||_i = \sum_{j=1}^{m} ||g_i(s - q_j)||_i + ||g_i(q_j) - a_j||_i = \sum_{j=1}^{m} ||g_i(s) - a_j||_i$$

that exists in each Banach space X_i.

(ii) implies (i):

Let $U \subset X$ be an absolutely convex 0-neighborhood. Suppose there exists a FTP defined on X_i. This further implies that there is a FTP defined on the range Img_i of g_i.

Hence, for each finite set $M = \{q_1, q_2, \ldots, q_m\} \subset X$ and 0-neighborhood $U \subset X$ with $r > 0$ there exists

$$s \in rU \text{ such that } ||g_i(s)||_i < r$$

and that

$$||g_i(s) - g_i(q_j)||_i < 1$$

whereby

$$\sum_{j=1}^{m} ||g_i(s) - g_i(q_j)||_i$$

is a FTP defined on Img_i for each X_i.

This implies that for

$$s \in rU \text{ and } s - q_j \in U,$$

$$\sum_{j=1}^{m} p_i(s - q_j)$$

where $p_i \in \mathcal{P}$ for all i exists and is defined on X. We can therefore conclude that (i) and (ii) are equivalent. $\square$

Proposition 4.1. *Let X be a Frechet space. Given a finite set $\{a_j\}_j$ in a Banach space X_n, let B_n be a bounded set for the FTP $\sum_{j=1}^{m} ||(s' - a_j)||$ on a Banach space X_n for each n, then given a finite set $\{q_j\}_j \subset X$, the set B for the FTP $\sum_{j=1}^{m} p_n(s - q_j)$ on X is bounded for all n.*

Proof. Define the FTP $T_{X_n} = \sum_{j=1}^{m} ||(s' - a_j)||$ on the Banach space X_n for each n associated with finite set $\{a_j\}_j$ for $s' \in X_n$. There is a gauge $p_{B_n}(s') = inf\{\lambda : s' \in \lambda B_n\}$, such that

$$T_{X_n} = \sum_{j=1}^{m} ||(s' - a_j)||_n = \sum_{j=1}^{m} p_{B_n}(s' - a_j)$$

on X_n for each n. Hence, by the gauge p_{B_n} on X_n and for any 0-neighbourhood V_n there exists a bounded convex neighbourhood of zero $B_n = \{s' : T(s') \leq T(t'), \quad t' \in X\}$ associated with the FTP $\sum_{j=1}^{m} p_{B_n}(s'-a_j)$ so that $B_n \subset \lambda V_n$ for each n, where $\lambda > 0$. See ([4], Proposition 2.7)

However, since the Frechet space $X = \bigcap X_n$ is dense in X_n and metrizable where $X_n = X/kerp_n$ for each gauge p_n on X is finite dimensional, then given a finite set $\{q_j\}_j \subset X$ and $s \in X$ there is a fixed collection of gauges on X for which the FTP $\sum_{j=1}^{m} p_n(s - q_j)$ is defined on the Frechet space X for all n, with $(j = 1, 2, 3, ...m \quad)$ by Theorem 4.1.

The Frechet space X carries a product topology, therefore, the set

$$B \subseteq \Pi_n B_n$$

is bounded in X.

Moreover, since every Frechet space is bornological, there exists an absolutely convex 0-neighbourhood U in X such that

$$B \subset \lambda U.$$

For the bounded set $B_n = \{s' : T(s') \leq T(t'), \quad t' \in X\}$ in X_n with $s' \in B_n$ for each n there exist $s \in X$ for which $s \in B$. □

The following proposition describes the properties of the FTP $N(s)$ in Frechet spaces.

Proposition 4.2. *Let X be a Frechet space with $\mathcal{P} = \{p_n\}_{n \in \mathbb{N}}$ defining its topology. Let $A = \{q_1, q_2, \ldots, q_m\} \subset X$. Then $N(s) = \sum_{i=1}^{m} p_n(s - q_i),\ s \in X$ satisfies the following*

(i) $N(s)$ is continuous on X;

(ii) $\lim_{p_n(s) \to \infty} N(s) = \infty$ and

(iii) $N(s)$ is convex on X.

Proof. (i) For distinct points $\{q_1, q_2, \ldots, q_m\} \subset X$ we have $q_i + U_n \ (n = 1, 2, \ldots)$ being a neighbourhood of q_i in X for each i where $U_n = \{s | p_n(s) \leq \varepsilon\}$ for some $\varepsilon > 0$. Therefore, $\{q_1, q_2, \ldots, q_m\} \subset \bigcup_{i=1}^{m}(q_i + \bigcap_{n=1} U_n)$ where $\varepsilon \bigcap_{n=1} U_n = \{s | sup_n(s) < \varepsilon\}$. Hence, for $s \in X$ then, $s \in q_i + U_n$ for each i. This implies that $s - q_i \in U_n$, for each i which also implies $p_n(s - q_i) \leq \varepsilon$ for all n therefore, we have

$$\sum_{i}^{m} |p_n(s) - p_n(q_i)| \leq \sum_{i=1}^{m} p_n(s - q_i) \leq m\varepsilon.$$

Hence $N(s)$ is continuous.

(ii) $N(s) = \sum_{i=1}^{m} p_n(s - q_i) \geq \sum_{i}^{m} |p_n(s) - p_n(q_i)| = |p_n(s) - \sum_{i=1}^{m} p_n(q_i)|$, this shows that as $p_n(s) \to \infty$, $\lim_{p_n(s) \to \infty} N(s) = \infty$.

(iii) For all $s, t \in X$ and $\beta \in [0, 1]$ we have $N(\beta s + (1 - \beta)t) = \sum_{i=1}^{m} p_n((\beta s + (1 - \beta)t) - q_i) \leq \sum_{i=1}^{m} p_n(\beta(s - q_i) + (1 - \beta)(t - q_i)) \leq \beta \sum_{i=1}^{m} p_n(s - q_i) + (1 - \beta) \sum_{i=1}^{m} p_n(t - q_i) = \beta N(s) + (1 - \beta)N(t)$. Hence, $N(s)$ is convex.

□

4.2 Existence results on collinear points

Propositions (3.3 and 3.4) are being generalized in this subsection for Frechet spaces.

The following definition is inspired by ([4] Definition 4.1) and Theorem 4.1. It also holds for Frechet spaces.

Definition 4.2. *Set of n distinct points $\{q_1, \cdots, q_n\}$ in a Frechet space X are said to be collinear if there exists two distinct elements s and t in X and $\{\lambda_i\}$ $i = 1, 2, ..., n$ such that $q_i = \lambda_i s + (1-\lambda_i)t$ $i = 1, 2, 3, ..., n$.*

Proposition 4.3. *Let X be a Frechet space and let $A = \{q_1, ..., q_{2k+1}\}$ be a set of odd collinear distinct points in X. Then, the subset of the set of minimizers for the FTP is $\{q_k, q_{k+1}\}$.*

Proof. The proof follows from Theorem (4.1) and the argument in the proof of [4 Proposition 4.2]. ☐

Proposition 4.4. *Let X be a Frechet space and let $A = \{q_1, ..., q_{2k}\}$ be a set of even collinaer distinct points in X. Then, the set of minimizers for the FTP is $\{q_{k+1}\}$.*

Proof. The proof follows from Theorem (4.1) and the argument in the proof of [4 Proposition 4.3]. ☐

4.3 Existence results on non collinear points

To generalize Proposition (3.1) for Frechet spaces, preliminary results in Propositions (4.5 and 4.6) and Theorem (4.3) are required.

The solution set for the FTP is defined as follows.

Definition 4.3. *Let X be a real Frechet space with $\mathcal{P} = \{p_n\}_{n\in\mathbb{N}}$ defining its topology. Let $A = \{q_1, q_2, \ldots, q_m\} \subset X$. Then, the set $O = \{s \in X | N(s) \leq N(t), t \in X\}$ will be called a set of minimizers for the FTP.*

Given any finite set $\{q_1, q_2, ...q_m\} \subset X$, the following proposition gives the properties of the set of minimizers O for FTP.

Proposition 4.5. *Let X be a Frechet space with $\mathcal{P} = \{p_n\}_{n\in\mathbb{N}}$ defining its topology. If $A = \{q_1, q_2, \ldots, q_m\} \subset X$. Then for $N(s)$ as given in Proposition 4.2, $O := \{s \in X | N(s) \leq N(t), t \in X\}$ is a convex, closed and bounded subset of X.*

Proof. From Proposition 4.2, $N(s)$ is convex. Hence, convexity of

$$O := \{s \in X | N(s) \leq N(t), t \in X\}$$

follows from the convexity of $N(s)$. We show next that O is closed. O is closed as the inverse image of a closed set.

Lastly, we need to show that $O = \{s | N(s) \leq N(t),\ t \in X\}$ is bounded.

From Proposition 4.1, we have

$$\sup_{s\in B} p_n(s) < \infty.$$

By definition $s \in O$, and since O is closed, it implies that $s \in B \subset O$. Hence, O as the closure of a bounded set B is bounded. ☐

Theorem 4.2 ([19]). *Let X be a lcs. X is reflexive if and only if it is barrelled and every bounded set is (relatively weakly compact) contained in a weakly compact set.*

Concerning weakly lower semicontinuous function on the weakly compact set K. The next proposition holds and then stated without proof.

Proposition 4.6. *Let X be a reflexive Frechet space and given a weakly compact set $K \subset X$, then the function $N : K \subset X \to [-\infty, \infty]$ is weakly lower semi continuous on the weakly compact set K.*

In the following theorem it is established that the FTP which is weakly lower semicontinuous for reflexive Frechet space X attains its bound in a weakly compact set in X.

Theorem 4.3. *If X is a reflexive Frechet spaces and given a weakly compact set $K \subset X$. Then $N : K \subset X \to [-\infty, \infty]$ which is weakly lower semi-continuous on K attains its bound in K.*

Proof. Let $O = \{s \in X | N(s) \leq N(t), t \in X\}$ be as we have in Proposition 4.5. Since O is bounded and closed. It is weakly compact by Theorem 4.2. Suppose $N : O \subset X \to [-\infty, \infty]$ is not bounded. This implies that for a sequence $\{s_j\} \subset O$, $N(s_j) \to -\infty$. But we have that O is weakly compact. With this, there exists a subsequence $\{s_{j_r}\}$ of $\{s_j\}$ $(j, j_r \in N)$ such that $s_{j_r} \to s$ weakly in O.

Since N is weakly lower semi continuous on O, $N(j_{j_r}) \overset{\text{weakly}}{\longrightarrow} N(s) = -\infty$. This is a contradiction since $0 \leq\ N(s) \leq N(t), t \in X$.

By definition $N(s) = \sum_{i=1}^{m} p_n(s - q_i)$ for all n, where $A = \{q_1, q_2, \ldots, q_m\} = \{q_i\}_i^m$ is finite in X.

Suppose $N(s_{j_r}) \overset{\text{weakly}}{\longrightarrow} N(s)$. That is,

$$\sum_{i=1}^{m} p_n(s_{j_r} - q_i) \longrightarrow \sum_{i=1}^{m} p_n(s - q_i) \tag{1}$$

for all n.
So also, since O is weakly compact, there exists a subsequence $\{s_{j_r}\}$ converging weakly to s. That is, $s_{j_r} \overset{\text{weakly}}{\longrightarrow} s$, which also implies

$$\sum_{i=1}^{m} p_n((s_{j_r} - s) - q_i) \leq m\varepsilon \tag{2}$$

for all n.

Hence, from (1) and (2), we have

$$|N(s_{j_r}) - N(s)| \leq N(s_{j_r} - s)$$

So also

$$|N(s) - N(s_{j_r})| \leq N(s_{j_r} - s)$$

Therefore,

$$|N(s)| \leq N(s_{j_r} - s) + N(s_{j_r})$$

i.e

$$|N(s)| \leq N(s_{j_r} + s_{j_r} - s)$$

Let $t = s_{nj} + s_{nj} - s$ in X.

Since $t \in X$ and by the condition of weak lower semicontinuity of N , $N(s) = inf_t N(t)$. Hence, $N(s) \leq\ N(t)$.

Therefore, N attains its bound in O. ☐

Theorem 4.4. *Let X be a reflexive Frechet space with $\mathcal{P} = \{p_n\}_{n \in \mathbb{N}}$ defining its topology and $A = \{q_i, \ldots, q_m\}$ a set of distinct points in X. Then, there exists a set of points that minimizes $N(s) = \sum_{i=1}^{m} p_n(s - q_i)$ on X.*

Proof. Let M be a bounded subsets of X. By the property of reflexivity, M is relatively weakly compact, whose closure is weakly compact. Hence, we define $O = \bar{M} = \{s \in X | N(s) \leq N(t)\}$. By Proposition 4.5 and Theorem 4.2, O is weakly compact. By Proposition 4.6, N is weakly lower semi continuous. By Theorem 4.3, $N(s)$ attains its bound which implies that it attains a minimum on O. ☐

4.4 Uniqueness of Torricellian point in Frechet spaces for non collinear points

Uniqueness of the minimizer of FTP in Frechet spaces is discussed in this subsection to generalizes Proposition (3.2). This is done by using strict convexity of the space and that of the functional.

Definition 4.4. *The lcs X is referred to as strictly convex if given a family $\mathcal{P} = \{p_i\}_{i\in I}$ of continuous seminorms for X and for every $s,t \in X$, $p_i(s+t) = p_i(s) + p_i(t)$ with $p_i(s) \neq p_i(t)$ then $p_i(t-\alpha s) = 0$ for some real number $\alpha > 0$ and $U_{(s,t)} \bigcap \ker p_i = \{0\}$. Where $U_{(s,t)}$ is the set that spans s and t.*

Remark 4.1. *Theorem 4.5 holds for Frechet space X. This recaptures the results of Ayinde and Osinuga [1], Theorem 3.3. Hence, the proof is omitted.*

Theorem 4.5. *Let X be a Frechet space (a projective limit of sequence of Banach spaces $\{X_i\}_{i\in\mathbb{N}}$). Then, X is strictly convex if and only if X_i is strictly convex.*

Theorem 4.6. *Let X be a Frechet space with $\mathcal{P} = \{p_n\}_{n\in\mathbb{N}}$ defining its topology. Suppose $\{q_1, \cdots, q_m\}$ is non-collinear set of points in X. If X is strictly convex, then $N(s)$ is strictly convex on X.*

Proof. Since $N(s)$ is convex by Proposition 4.2 then, we can write

$$N(\lambda s + (1-\lambda)t) \leq \lambda N(s) + (1-\lambda)N(t)$$

where $0 \leq \lambda \leq 1$ and $s,t \in X$. We shall show this by contradiction. Since N is called strictly convex if $N(\lambda s + (1-\lambda)t) < \lambda N(s) + (1-\lambda)N(t)$. In view of this, let assume that N is not strictly convex i.e. $N(\lambda s + (1-\lambda)t) = \lambda N(s) + (1-\lambda)N(t)$

Hence,

$$\sum_{i=1}^{m} p_n(\lambda(s-q_i) + (1-\lambda)(t-q_i)) = |\lambda| \sum_{i=1}^{m} p_n(s-q_i) + |1-\lambda| \sum_{i=1}^{m} p_n(t-q_i)$$

for all n. Since the terms are non-negative, we then have

$$p_n(\lambda(s-q_i) + (1-\lambda)(t-q_i)) = p_n(\lambda(s-q_i)) + p_n((1-\lambda)(t-q_i))$$

for $i = 1, 2, \cdots, m$

Since X is strictly convex. That is, $p_n(s+t) = p_n(s) + p_n(t)$ implies that $p_n(s) \neq p_n(t)$ such that $p_n(t-\beta s) = 0$ for a $\beta > 0$. If $\lambda(s-q_i) = s_1$ and $(1-\lambda)(t-q_i) = s_2$. There exist $\beta > 0$ such that $p_n(s_1 - \beta s_2) = 0$ This implies that

$s_1 - \beta s_2 \in kerp$ and $s_1 - \beta s_2 \in U_{(s_1,s_2)}$.
Hence,

$$s_1 - \beta s_2 \in U_{(s_1,s_2)} \bigcap \ker p = \{0\}$$

That is

$$s_1 - \beta s_2 = (\lambda s - \lambda q_i - \beta t + \beta q_i + \beta\lambda t - \beta\lambda q_i) = 0$$

i.e.,

$$(-\lambda q_i + \beta q_i - \beta\lambda q_i + \lambda s - \beta t + \beta\lambda t) = 0$$

$$(q_i(-\lambda + \beta(1-\lambda)) + \lambda s - \beta(1-\lambda)t) = 0$$

Hence, $q_i(-\lambda + \beta(1-\lambda)) = \beta(1-\lambda)t - \lambda s$

$$q_i = \frac{\beta(1-\lambda)}{\beta(1-\lambda)-\lambda}t - \frac{\lambda}{\beta(1-\lambda)-\lambda}s$$

This implies that q_i lie on the line segment that connects s and t. This is a contradiction to $\{q_1, \cdots, q_m\}$ being non-collinear. So also suppose $s, t \neq q_i$ for all $i = 1, 2, \cdots, m$ and $\beta = \frac{\lambda}{1-\lambda}$

This translates to

$$p_n(q_i(-\lambda + \frac{\lambda}{1-\lambda}(1-\lambda)) + \lambda s - \frac{\lambda}{1-\lambda}(1-\lambda)t) = 0$$

i.e., $p_n(\lambda s - \lambda t) = 0$, this gives $|\lambda| p(s-t) = 0$

$$i.e. \;\; p_n(s-t) = 0.$$

Hence, since $|p_n(s) - p_n(t)| \leq p_n(s-t) = 0$. This implies that $|p_n(s) - p_n(t)| = 0$. Therefore, $p_n(s) = p_n(t)$

This is also a contradiction to the assumption that $p_n(s) \neq p_n(t)$ and couple with the first contradiction, we can conclude that $N(s)$ is strictly convex. □

Theorem 4.7. *Let X be a Frechet space with $\mathcal{P} = \{p_n\}_{n\in\mathbb{N}}$ defining its topology. Let $\{q_i, \cdots, q_m\} \subset X$ be non-collinear set of points. If X is strictly convex then $O = \{s \mid N(s) \leq N(t)\}$ contains a unique element.*

Proof. Let $s, t \in O$. This implies by definition that $N(s) = N(t)$.

Since X is strictly convex which by Theorem 4.6 implies that $N(s)$ is strictly convex. Hence

$$N(\lambda s + (1-\lambda)t) < \lambda N(s) + (1-\lambda)N(s) = N(s)$$

$$\lambda \in [0,1], \qquad let \;\; \lambda = \frac{1}{2}$$

$$N(\frac{s}{2} + \frac{t}{2}) < \frac{1}{2}N(s) + \frac{1}{2}N(t) = N(s)$$

$$i.e. \;\; N(\frac{s+t}{2}) + \frac{N(s)+N(t)}{2} = N(s)$$

This is a contradiction, hence, $s = t$ which implies that O contain a unique element. □

5 Conclusions

This paper considered Fermat-Torricelli problem on Frechet spaces as a result of motivation coming from established results on Fermat-Torricelli problem in normed linear spaces and Banach spaces visited in light of convex analysis and various norms. This generalization was made by the use of seminorms as gauges. It was implied that many results for normed linear and Banach spaces respectively were carried over to this generalized settings and many more in that direction. For a reflexive Frechet space, it was established that the set of minimizers is weakly compact. This helps to show the existence results for non collinear points. Convexity criteria were later employed to establish the uniqueness of the minimizer in the case of non collinear points.

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