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A Result on an Integral Involving Product of Two Generalized Hypergeometric Functions and Its Applications

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Abstract: *The aim of this research note is to provide an interesting integral involving product of generalized hypergeometric functions. This is achieved with the help of a known integral involving hypergeometric function available in the literature. Several results obtained earlier by Harsh, et al. [5] follow special cases and some applicable particular examples of our main findings.*

Keywords: Hypergeometric function, Generalized hypergeometric function, Watson theorem, Definite integral

1 Introduction and Results Required

In order to justify our work, we must quote Sylvester [14] ‘‘It seems that every mathematician ascending the heights of mathematical achievement is expected, at some stage of their journey, to pause and contribute a new definite integral or two, adding to the collective body of knowledge.” This research paper has significant mathematical importance, especially in the fields of applied mathematics, mathematical physics, and engineering. Generalized hypergeometric functions are fundamental in mathematical analysis and often arise in solutions to complex differential equations [1, 6]. Results that involve integrals of these functions, particularly products of two such functions, deepen our understanding of their properties and interrelations [7, 10, 11]. Generalized hypergeometric functions are key components in mathematical analysis, with applications across fields like physics, engineering, and statistics. The research work potentially provides new results that enrich the theory of hypergeometric functions by exploring integrals involving products of these functions, thereby broadening the scope of known results in the field. Integrals involving products of generalized hypergeometric functions are challenging to evaluate, and closed-form solutions are often rare. Thus, the manuscript not only enriches the mathematical literature with a valuable closed-form integral but also offers an example of a complex evaluation, which can be referenced in future mathematical, scientific, and engineering research [8, 12, 13].

Let us start by recalling that the natural extension of Gauss's hypergeometric function ${}_2F_1$ is known as the generalized hypergeometric function ${}_pF_q$ defined by[1, 3, 12, 13]

$$ {}_pF_q\left[\begin{matrix} s_1, & \ldots, & s_p \\ & & & ; & z \\ t_1, & \ldots, & t_q \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(s_1)_n \cdots (s_p)_n}{(t_1)_n \cdots (t_q)_n} \frac{z^n}{n!} \tag{1} $$

where Pochhammer symbol $(s)_n$ is defined (for $s \in \mathbb{C}$) by

$$ \begin{aligned} (s)_n &= \frac{\Gamma(s+n)}{\Gamma(s} \qquad (s \in \mathbb{C}\backslash \mathbb{Z}_0^-) \\ &= \begin{cases} s.(s+1)...(s+n-1) & ; n \in \mathbb{N}, \\ 1 & ; n = 0, \end{cases} \end{aligned} \tag{2} $$

$$ \Gamma(s+1) = s\Gamma(s), \quad \Gamma(s)\Gamma(1-s) = \frac{\pi}{sin\pi s} \tag{3} $$

Here, $\Gamma(s)$ denotes the well-known Gamma function. By convention, an empty product is taken to equal one. We assume that the variable z and the parameters $s_1, \ldots, s_p$ in the numerator, as well as $t_1, \ldots, t_q$ in the denominator, can all assume complex values, provided there are no zeros in the denominator of equation (1), that is

$$(t_j \in \ \mathbb{C}\backslash\mathbb{Z}_0^-; j = 1, \ldots, q).$$

Let $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ represent the sets of complex numbers, integers, and positive integers, respectively, in both cases, and moreover, let

$$\mathbb{N}_0 = \mathbb{N} \cup 0 \ \text{ and } \ \mathbb{Z}_0^- := \mathbb{Z}\backslash\mathbb{N}.$$

For more information on ${}_pF_q$ including its convergence, special cases, and useful limits, we refer to standard texts [1, 3, 12, 13]. Summations of generalized hypergeometric functions in terms of gamma functions—such as ${}_2F_1$ with arguments like 1, $\frac{1}{2}$, and -1 are particularly valuable in applications. Classical summation theorems by Gauss, Kummer, Bailey, Watson, Dixon, Whipple, and Saalschütz play a central role in theory and applications; for further insights, see Bailey's work [2]. Below, we highlight Watson's classical summation theorem [1, 3, 12, 13]

$${}_3F_2\left[\begin{array}{ccc} a, & b, & c \\ \frac{a+b+1}{2}, & 2c & \end{array}; 1\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)}, \tag{4}$$

where $Re(2c - a - b) > -1$.

Using (4), we can compute the following integral involving the hypergeometric function, as documented in references [4, 5], namely:

$$\begin{aligned}&\int_0^{\pi/2} e^{2ci\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}\ {}_2F_1\left[\begin{array}{cc} a, & b \\ \frac{a+b+1}{2} & \end{array}; \ e^{i\theta}\cos\theta\right] d\theta \\ &= e^{\frac{i\pi c}{2}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(c)\Gamma(c)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(2c)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)},\end{aligned} \tag{5}$$

provided that $Re(c) > 0$ and $Re(2c - a - b) > -1$.

In this work, an intriguing integral involving the product of two generalized hypergeometric functions has been evaluated in terms of the gamma function, using the known integral (5). Additionally, several notable special cases and some particular applicable examples are provided. For similar results refer [9, 10, 11].

2 Main Integral Formula

In this section, we will evaluate an integral involving the product of two generalized hypergeometric functions, as stated in the following theorem.

Theorem 2.1. *For $Re(d) > 0$ and $Re(2d-a-b) > -1$, the following result holds true;*

$$\int_0^{\pi/2} e^{2di\theta}(sin\theta)^{d-1}(cos\theta)^{d-1}\ {}_2F_1\left[\begin{array}{c} a, \quad b \\ \frac{a+b+1}{2} \end{array}; e^{i\theta}\cos\theta\right]$$
$$\times\ {}_2F_2\left[\begin{array}{c} c-\frac{a}{2}+\frac{1}{2}, \quad c-\frac{b}{2}+\frac{1}{2} \\ c, \quad c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2} \end{array}; -4ie^{2i\theta}\sin\theta\cos\theta)\right] d\theta$$
$$= e^{\frac{i\pi d}{2}} \frac{\sqrt{\pi}\Gamma(d)\Gamma(d)\Gamma\left(d+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(d-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(2d)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(d-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(d-\frac{b}{2}+\frac{1}{2}\right)}$$
$$\times\ {}_4F_4\left[\begin{array}{c} d, \quad c-\frac{a}{2}+\frac{1}{2}, \quad c-\frac{b}{2}+\frac{1}{2}, \quad d-\frac{a}{2}-\frac{b}{2}+\frac{1}{2} \\ c, \quad d-\frac{a}{2}+\frac{1}{2}, \quad d-\frac{b}{2}+\frac{1}{2}, \quad c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2} \end{array}; 1\right]. \tag{6}$$

Proof. To evaluate integral (6), we proceed as follows. Letting I represent the left-hand side of (6), we have

$$I = \int_0^{\pi/2} e^{2di\theta}(\sin\theta)^{d-1}(\cos\theta)^{d-1}\ {}_2F_1\left[\begin{array}{c} a, \quad b \\ \frac{a+b+1}{2} \end{array}; e^{i\theta}\cos\theta\right]$$
$$\times\ {}_2F_2\left[\begin{array}{c} c-\frac{a}{2}+\frac{1}{2}, \quad c-\frac{b}{2}+\frac{1}{2} \\ c, \quad c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2} \end{array}; -4ie^{2i\theta}\sin\theta cos\theta)\right] d\theta.$$

Now, by expressing ${}_2F_2$ as a series and interchanging the order of integration and summation, justified by the uniform convergence of the series involved, we obtain

$$I = \sum_{n=0}^{\infty} \frac{\left(c-\frac{a}{2}+\frac{1}{2}\right)_n \left(c-\frac{b}{2}+\frac{1}{2}\right)_n 2^n}{(c)_n \left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)_n n!}(-i)^n(2)^{2n}$$
$$\times \int_0^{\pi/2} e^{2(d+n)i\theta}(\sin\theta)^{d+n-1}(\cos\theta)^{d+n-1}\ {}_2F_1\left[\begin{array}{c} a, \quad b \\ \frac{a}{2}+\frac{b}{2}+\frac{1}{2} \end{array}; e^{i\theta}\cos\theta\right] d\theta.$$

By evaluating the integral using result (5) and applying result (2), we obtain, after some simplification:

$$I = e^{\frac{i\pi d}{2}} \frac{\Gamma(\frac{1}{2})\Gamma(d)\Gamma(d)\Gamma\left(d+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(d-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(2d)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(d-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(d-\frac{b}{2}+\frac{1}{2}\right)}$$

$$\times \sum_{n=0}^{\infty} \frac{(d)_n \left(c-\frac{a}{2}+\frac{1}{2}\right)_n \left(c-\frac{b}{2}+\frac{1}{2}\right)_n (d-\frac{a}{2}+\frac{b}{2}+\frac{1}{2})_n}{(c)_n \left(d-\frac{a}{2}+\frac{1}{2}\right)_n \left(d-\frac{b}{2}+\frac{1}{2}\right)_n \left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)_n}.$$

Finally using (1), the RHS of (6) is obtained. Thus the theorem is proved. □

3 Special Cases

This section covers notable cases of the main integral represented by (2.1), as presented in the corollaries below.

Corollary 3.1

In (2.1), if we set $b=-2n$ and replace a with $a+2n$, where n is zero or a positive integer, we obtain the following result:

$$\begin{aligned}&\int_0^{\pi/2} e^{2di\theta}(\sin\theta)^{d-1}(\cos\theta)^{d-1}\ {}_2F_1\left[\begin{matrix}-2n, & a+2n \\ & \\ \frac{a+1}{2} & \end{matrix};\ e^{i\theta}\cos\theta\right]\\&\times\ {}_2F_2\left[\begin{matrix}c+n+\frac{1}{2}, & c-\frac{a}{2}+\frac{1}{2}-n \\ & \\ c, & c-\frac{a}{2}+\frac{1}{2}\end{matrix};\ -4ie^{2i\theta}\sin\theta\cos\theta)\right]d\theta\\&=e^{\frac{i\pi d}{2}}\frac{\Gamma(d)\Gamma(d)\left(\frac{1}{2}\right)_n\left(\frac{1}{2}+\frac{a}{2}-d\right)_n}{\Gamma(2d)\left(d+\frac{1}{2})_n\right)\left(\frac{a+1}{2}\right)_n}\\&\times\ {}_4F_4\left[\begin{matrix}d, & c+n+\frac{1}{2}, & c-n-\frac{a}{2}+\frac{1}{2}, & d-\frac{a}{2}+\frac{1}{2} \\ & & & \\ c, & d+n+\frac{1}{2}, & d-n-\frac{a}{2}+\frac{1}{2}, & c-\frac{a}{2}+\frac{1}{2}\end{matrix};1\right].\end{aligned} \tag{7}$$

Corollary 3.2

In (2.1), if we set $b=-2n-1$ and replace a with $a+2n+1$, where n is zero or a positive integer, we obtain the following result:

$$\begin{aligned}&\int_0^{\pi/2} e^{2di\theta}(\sin\theta)^{d-1}(\cos\theta)^{d-1}\ {}_2F_1\left[\begin{matrix}-2n-1, & a+2n+1 \\ & \\ \frac{a+1}{2} & \end{matrix};\ e^{i\theta}\cos\theta\right]\\&\times\ {}_2F_2\left[\begin{matrix}c+n+1, & c-\frac{a}{2}-n \\ & \\ c, & c-\frac{a}{2}+\frac{1}{2}\end{matrix};\ -4ie^{2i\theta}\sin\theta\cos\theta)\right]d\theta\\&=0.\end{aligned} \tag{8}$$

This is a typical example.

Corollary 3.3

In (2.1), if we set $a=b=\frac{1}{2}$ and utilizing the established result [4]

$${}_2F_1\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2} \\ & \\ 1 & \end{matrix};\ x\right]=\frac{2}{\pi}K(\sqrt{x}),$$

where $K(k)$ refers to the well-known elliptic integral of the first kind, defined by.

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{1-k^2\sin^2 t}}.$$

Then, we obtain the following interesting result:

$$\begin{aligned}&\int_0^{\pi/2} e^{2di\theta}(\sin\theta)^{d-1}(\cos\theta)^{d-1}\ K(\sqrt{e^{i\theta}\cos\theta})\ {}_2F_2\left[\begin{matrix}c+\frac{1}{4}, & c+\frac{1}{4} \\ & \\ c, & c\end{matrix};\ -4ie^{2i\theta}\sin\theta\cos\theta\right]d\theta\\&=e^{\frac{i\pi d}{2}}\frac{\pi^{\frac{3}{2}}}{2}\frac{\Gamma^3(d)\Gamma\left(d+\frac{1}{2}\right)}{\Gamma(2d)\Gamma^2\left(\frac{3}{4}\right)\Gamma^2\left(d+\frac{1}{4}\right)}\times_4F_4\left[\begin{matrix}d, & d, & c+\frac{1}{4}, & c+\frac{1}{4} \\ & & & \\ c, & c, & d+\frac{1}{4} & d+\frac{1}{4}\end{matrix};1\right],\end{aligned} \tag{9}$$

provided $Re(d) > 0$.

Corollary 3.4
In (6), if we set $a = b = 1$ and utilizing the established result [4]

$$ {}_2F_1\left[\begin{matrix} 1, & 1 \\ & \\ \frac{3}{2} & \end{matrix} ; x\right] = \frac{\sin^{-1}(\sqrt{x})}{\sqrt{x(1-x)}}. $$

Then, we obtain the following interesting result:

$$ \int_0^{\pi/2} e^{2di\theta}(\sin\theta)^{d-1}(\cos\theta)^{d-1} \frac{\sin^{-1}(\sqrt{e^{i\theta}\cos\theta})}{\sqrt{e^{i\theta}\cos\theta(1-e^{i\theta}\cos\theta)}} \times {}_1F_1\left[\begin{matrix} c \\ \\ c-\frac{1}{2} \end{matrix} ; 1\right] d\theta = e^{\frac{i\pi d}{2}} \frac{\pi}{2} \frac{\Gamma\left(d-\frac{1}{2}\right)\Gamma\left(d+\frac{1}{2}\right)}{\Gamma(2d)} \times {}_2F_2\left[\begin{matrix} c, & d-\frac{1}{2} \\ & \\ d, & c-\frac{1}{2} \end{matrix} ; 1\right], \tag{10} $$

provided $Re(d) > \frac{1}{2}$.

Corollary 3.5

In (6), if we set $b = -a$ and utilizing the established result [4], then

$$ \int_0^{\pi/2} e^{2di\theta}(\sin\theta)^{d-1}(\cos\theta)^{d-1} \cos(2a\sin^{-1}\sqrt{e^{i\theta}\cos\theta})\, {}_2F_2\left[\begin{matrix} c-\frac{a}{2}+\frac{1}{2}, & c+\frac{a}{2}+\frac{1}{2} \\ & \\ c, & c+\frac{1}{2} \end{matrix} ; -4ie^{2i\theta}\sin\theta\cos\theta\right] d\theta $$
$$ = e^{\frac{i\pi d}{2}} \frac{\pi\Gamma^2(d)\Gamma^2\left(d+\frac{1}{2}\right)}{\Gamma(2d)\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)\Gamma\left(d+\frac{1}{2}+\frac{a}{2}\right)\Gamma\left(d+\frac{1}{2}-\frac{a}{2}\right)} \times {}_4F_4\left[\begin{matrix} d, & d+\frac{1}{2}, & c-\frac{a}{2}+\frac{1}{2}, & c+\frac{a}{2}+\frac{1}{2} \\ & & & \\ c, & c+\frac{1}{2}, & d-\frac{a}{2}+\frac{1}{2}, & d+\frac{a}{2}+\frac{1}{2} \end{matrix} ;1\right], \tag{11} $$

provided , $Re(c) > 0, Re(d) > 0$. Finally in (6) and (7) to (11) if $d = c$, we get the following results in more compact form:

$$ \int_0^{\pi/2} e^{2ci\theta}(\sin\theta)^{c-1}(cos\theta)^{c-1}\, {}_2F_1\left[\begin{matrix} a, & b \\ & \\ \frac{a+b+1}{2} & \end{matrix} ; e^{i\theta}\cos\theta\right] \times {}_2F_2\left[\begin{matrix} c-\frac{a}{2}+\frac{1}{2}, & c-\frac{b}{2}+\frac{1}{2} \\ & \\ c, & c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2} \end{matrix} ; -4ie^{2i\theta}\sin\theta\cos\theta\right] d\theta $$
$$ = e^{\frac{i\pi c}{2}} \frac{e\sqrt{\pi}\Gamma(c)\Gamma(c)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}+\frac{1}{2}\right)}{\Gamma(2c)\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)}, \tag{12} $$

and

$$\int_0^{\pi/2} e^{2ci\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}\ {}_2F_1\left[\begin{array}{c} -2n, \quad a+2n \\ \\ \frac{a+1}{2} \end{array} ;\quad e^{i\theta}\cos\theta\right]$$

$$\times\ {}_2F_2\left[\begin{array}{cc} c+n+\frac{1}{2}, & c-\frac{a}{2}+\frac{1}{2}-n \\ \\ c, & c-\frac{a}{2}+\frac{1}{2} \end{array} ;\quad -4ie^{2ci\theta}\sin\theta\cos\theta\right] d\theta$$

$$= e^{\frac{i\pi c}{2}} \frac{e\Gamma(c)\Gamma(c)(\frac{1}{2}+\frac{a}{2}-c)_n}{\Gamma(2c)(c+\frac{1}{2})_n(\frac{a+1}{2})_n}, \tag{13}$$

and

$$\int_0^{\pi/2} e^{2ci\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}{}_2F_1\left[\begin{array}{c} -2n-1, \quad a+2n+1 \\ \\ \frac{a+1}{2}, \end{array} ; e^{i\theta}\cos\theta\right]$$

$$\times_2 F_2\left[\begin{array}{cc} c+n+1, & c-\frac{a}{2}-n, \\ \\ c, & c-\frac{a}{2}+\frac{1}{2} \end{array} ; -4ie^{2i\theta}\sin\theta\cos\theta\right] d\theta = 0, \tag{14}$$

and

$$\int_0^{\pi/2} e^{2ci\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}K(\sqrt{e^{i\theta}\cos\theta})_2F_2\left[\begin{array}{cc} c+\frac{1}{4}, & a+\frac{1}{4} \\ \\ c, & c \end{array} ; -4ie^{2i\theta}\sin\theta\cos\theta\right] d\theta$$

$$= e^{\frac{i\pi c}{2}} \frac{e}{2}\pi^{3/2}\frac{\Gamma^3(c)\Gamma(c+\frac{1}{2})}{\Gamma(2c)\Gamma^2(\frac{3}{4})\Gamma^2(c+\frac{1}{4})}, \tag{15}$$

and

$$\int_0^{\pi/2} e^{2ci\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}.\frac{sin^{-1}(\sqrt{e^{i\theta}\cos\theta})}{\sqrt{e^{i\theta}\cos\theta(1-e^{i\theta}cos\theta)}}d\theta$$

$$= e^{\frac{i\pi c}{2}} \frac{e\pi}{2}\frac{\Gamma(c-\frac{1}{2})\Gamma(c+\frac{1}{2})}{\Gamma(2c)} \tag{16}$$

provided $Re(c) > \frac{1}{2}$ and finally

$$\int_0^{\pi/2} e^{2ci\theta}(\sin\theta)^{c-1}(\cos\theta)^{c-1}cos(2a\sin^{-1}\sqrt{e^{i\theta}\cos\theta})_2F_2\left[\begin{array}{cc} c-\frac{a}{2}+\frac{1}{2}, & c+\frac{a}{2}+\frac{1}{2} \\ \\ c, & c+\frac{1}{2} \end{array} ; -4ie^{2i\theta}\sin\theta\cos\theta)\right] d\theta$$

$$= e^{\frac{i\pi c}{2}} \frac{\pi e\Gamma^2(c)\Gamma^2(c+\frac{1}{2})}{\Gamma(2c)\Gamma(\frac{1}{2}-\frac{a}{2})\Gamma(c+\frac{1}{2}+\frac{a}{2})\Gamma(c+\frac{1}{2}-\frac{a}{2})\Gamma(\frac{1}{2}+\frac{a}{2})}, \tag{17}$$

provided $Re(c) > 0$. Similarly other result can be obtained In the results (7) to (11), if we set $d = c$, we get the known results due to Harsh et al. [5].

4 Particular Examples

In the main integral theorem and its special cases, substituting specific numerical values for the parameters results in several significant integrals. These integrals are widely applicable across various practical fields.

A few key examples are presented below.

1. Putting $a = b = \frac{1}{2}, c = 2$ in (12) then

$$\int_0^{\pi/2} e^{4i\theta} \sin\theta \; \cos\theta \; {}_2F_1 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ & \\ 1 & \end{matrix} ; \; e^{i\theta}\cos\theta \right] \times \; {}_2F_2 \left[\begin{matrix} \frac{9}{4}, & \frac{9}{4} \\ & \\ 2, & 2 \end{matrix} ; \; -4ie^{2i\theta} \sin\theta \; \cos\theta \right] d\theta$$

$$= e^{i\pi} e\sqrt{\pi} \frac{\Gamma(2)\Gamma(2)\Gamma(\frac{5}{2})\Gamma(1)\Gamma(2)}{\Gamma(4)\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{9}{4}\right)\Gamma\left(\frac{9}{4}\right)} = -\frac{16e}{25\pi} \quad [using\ (4)]$$

2. Putting $a = \frac{1}{2}, n = 1$ in (13) then

$$\int_0^{\pi/2} e^{4i\theta} \sin\theta \; \cos\theta \; {}_2F_1 \left[\begin{matrix} -2, & \frac{5}{2} \\ & \\ \frac{3}{4} & \end{matrix} ; \; e^{i\theta}\cos\theta \right] \times \; {}_2F_2 \left[\begin{matrix} \frac{7}{2}, & \frac{5}{4} \\ & \\ 2, & \frac{9}{4} \end{matrix} ; \; -4ie^{4i\theta} \sin\theta \; \cos\theta \right] d\theta$$

$$= e^{i\pi} e \frac{\Gamma(2)\Gamma(2)(\frac{-5}{4})_1}{\Gamma(4)(\frac{5}{2})_1(\frac{3}{4})_1} = \frac{e}{9} \quad [using\ (4)] \tag{18}$$

3. Putting $a = \frac{1}{2}, c = 2$ in (15) then

$$\int_0^{\pi/2} e^{4i\theta} \sin\theta \; \cos\theta \; K(\sqrt{e^{i\theta}\cos\theta}) \; {}_2F_2 \left[\begin{matrix} \frac{9}{4}, & \frac{3}{4} \\ & \\ 2, & 2 \end{matrix} ; -4ie^{2i\theta} \sin\theta \; \cos\theta \right] d\theta$$

$$= e^{i\pi} \frac{e\pi^{3/2}}{2} \frac{\Gamma^3(2)\Gamma(\frac{5}{2})}{\Gamma(4)\Gamma^2(\frac{3}{4})\Gamma^2(\frac{9}{4})} = \frac{-8e}{25} \quad [using\ (1.3)]$$

4. Putting $c = 2$ in (16), we obtain

$$\int_0^{\pi/2} e^{4i\theta} \sin\theta \; \cos\theta \; \frac{\sin^{-1}(\sqrt{e^{i\theta}\cos\theta})}{\sqrt{e^{i\theta}\cos\theta(1 - e^{i\theta}\cos\theta)}} d\theta = e^{i\pi} \frac{e\pi}{2} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(4)} = \frac{-e\pi^2}{32} \quad [using\ (4)] \tag{19}$$

5. Putting $a = \frac{1}{2}, c = 2$ in (17), we obtain

$$\int_0^{\pi/2} e^{4i\theta} \sin\theta \; \cos\theta \; cos(2a \sin^{-1}\sqrt{e^{i\theta}\cos\theta}) \; {}_2F_2 \left[\begin{matrix} \frac{9}{4}, & \frac{11}{4} \\ & \\ 2, & \frac{5}{2} \end{matrix} ; -4ie^{2i\theta} \sin\theta \cos\theta) \right] d\theta$$

$$= e^{i\pi} \pi e \frac{\Gamma^2(2)\Gamma^2(\frac{5}{2})}{\Gamma(4)\Gamma(\frac{1}{4})\Gamma(\frac{1}{4})\Gamma(\frac{9}{4})\Gamma(\frac{3}{4})} = \frac{-4e}{35} \quad [using\ (4)]$$

5 Conclusion

In this research, we have obtained an integral involving the product of two generalized hypergeometric functions in terms of Gamma function. Also we have an integral involving five corollaries including the special cases for a and b in the integral of product of two hypergeometric functions. The study of integrals involving products of generalized hypergeometric functions is rich with potential future research directions. The integral representations of products of hypergeometric functions in higher dimensions, which may have applications in multidimensional statistical distributions and mathematical physics.

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