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Blow up of Solutions for the Logarithmic Higher-Order Kirchhoff-Type Equation with Variable Exponent

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Abstract: *In this work, we investigate the logarithmic higher-order Kirchhoff-type equation with variable exponents as follows*

$$\theta_{tt} + \mathcal{M}\left(\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2\right)\mathcal{P}\theta + |\theta_t|^{p(x)-2}\theta_t = |\theta|^{q(x)-2}\theta\ln\theta.$$

We proved that under suitable conditions on the initial data, a finite-time blow up result for solutions with negative initial energy.

Keywords: Blow up, Kirchhoff-type equation, Variable exponents

1 Introduction

In this paper, we investigate the following problem:

$$\begin{cases} \theta_{tt} + \mathcal{M}\left(\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2\right)\mathcal{P}\theta + |\theta_t|^{p(x)-2}\theta_t = |\theta|^{q(x)-2}\theta\ln\theta, & \Omega\times(0,T), \\ \theta(x,t) = \frac{\partial}{\partial v}\theta(x,t) = 0, & \partial\Omega\times(0,T), \\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x) & x\in\Omega, \end{cases} \tag{1}$$

where $\mathcal{P} = (-\Delta)^m$, $m \geq 1$ is a natural number. $\Omega \subset R^n$ $(n \in N^+)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\mathcal{M}(\lambda) = s_1 + s_2\lambda^\gamma$ and $s_1,\ s_2 \geq 0$, $\gamma \geq 1$. $p(\cdot)$ and $q(\cdot)$ are given measurable functions on Ω, satisfying

$$\begin{cases} 2 \leq p_1 \leq p(x) \leq p_2 \leq p_*, \\ 2 \leq q_1 \leq q(x) \leq q_2 \leq q_* \end{cases} \tag{2}$$

here

$$\begin{cases} p_1 = ess\ \inf_{x\in\Omega} p(x),\ p_2 = ess\sup_{x\in\Omega}\ p(x) \\ q_1 = ess\ \inf_{x\in\Omega} q(x),\ q_2 = ess\sup_{x\in\Omega}\ q(x) \end{cases} \tag{3}$$

and

$$\begin{cases} 2 < p_* < \infty & \text{if } n \leq 2m, \\ 2 < p_* < \frac{2n}{n-2m} & \text{if } n > 2m, \end{cases} \tag{4}$$

also satisfying the log-Hölder continuity condition:

$$|p(x) - p(y)| \leq \frac{A}{\ln\left|\frac{1}{x-y}\right|}, \tag{5}$$

for all $x, y \in \Omega$ with $|x-y| < \delta$, $0 < \delta < 1$, $A > 0$.

In recent years, these problems appear in many modern physical and engineering models such as electro-rheological fluids, fluids with temperature dependent viscocity, filtration processes through a porous media, image processing and thermorheological fluids and others, which required modeling with non-standard [3, 12]. Before going any further, some imporant works in the literature are reviewed.
Tebba et al. [11] investigated a nonlinear damped wave equation given by:

$$\theta_{tt} - \Delta\theta - \Delta\theta_{tt} + a|\theta_t|^{m(x)-2}\theta_t = b|\theta|^{p(x)-2}\theta,$$

under appropriate assumptions on the variable exponents, they demonstrated the existence of a unique weak solution using the Faedo-Galerkin method. They also proved the finite time blow-up of solutions.

In the study by Ouaoua et al. [8], they investigated the following equation:

$$\theta_{tt} + \Delta^2\theta - \Delta\theta + |\theta_t|^{m(x)-2}\,\theta_t = |\theta|^{r(x)-2}\,\theta,$$

they demonstrated the local existence and also proved that the local solution is global.
In the study by Hamadouche [6], he investigated the following nonlinear Petrovsky equation:

$$\theta_{tt} + \Delta^2\theta + a\,|\theta_t|^{m(\cdot)-2}\,\theta_t = b\,|\theta|^{p(\cdot)-2},$$

by utilizing the Faedo-Galerkin method, the author established the existence of a unique weak solution for variable exponents m and p under suitable assumptions, and also obtained the blow-up result with negative initial energy.

Antontsev et al. [2] studied the following wave equation

$$\theta_{tt} + \Delta^2\theta - M\left(\|\nabla\theta\|^2\right)\Delta\theta + |\theta_t|^{p(x)-2}\,\theta_t = |\theta|^{q(x)-2}\,\theta.$$

By virtue of the Faedo-Galerkin method, they proved the local existence of the solution.

Liao et al. [7] studied following equation

$$\theta_{tt} + \Delta^2\theta - M\left(\|\nabla\theta\|^2\right)\Delta\theta - \Delta\theta_t + |\theta_t|^{m(x)-2}\,\theta_t = |\theta|^{p(x)-2}\,\theta, \tag{6}$$

they studied blow-up will happen for arbitrarily high initial energy.

Antontsev et al. [1] considered the Petrovsky equation with strong damping term of the form

$$\theta_{tt} + \Delta^2\theta - \Delta\theta_t + |\theta_t|^{p(x)-2}\,\theta_t = |\theta|^{q(x)-2}\,\theta.$$

They proved the local weak solutions and global nonexistence.

Pişkin [9] proved the nonexistence of solution of the following equation

$$\theta_{tt} - M\left(\|\nabla\theta\|^2\right)\Delta\theta + |\theta_t|^{p(x)-2}\,\theta_t = |\theta|^{q(x)-2}\,\theta.$$

Rahmoune [10] studied the following wave equation

$$\theta_{tt} - \Delta\theta + |\theta_t|^{m(x)-2}\,\theta_t = |\theta|^{p(x)-2}\,\theta\ln\theta,$$

they proved the local existence and blow up.

Dinç et al. [5] investigated the following Kirchhoff-type equation with a variable exponent:

$$\theta_{tt} - M\left(\|\nabla\theta\|_p^p\right)\Delta_p\theta + |\theta_t|^{r(x)-2}\,\theta_t = |\theta|^{q(x)-2}\,\theta.$$

Under suitable conditions, they established an upper bound for the blow-up time.

Motivated by the above studies, we proved to blow up the variable-exponent high-order logarithmic Kirchhoff-type equation.

This work is organized as follows. In the next part, we introduce preliminary details about variable exponent Lebesgue and Sobolev spaces. Moreover, we introduce important lemmas and assumptions. In Part 3, we prove our results by demonstrating that there is a finite-time blow-up for initial data with negative initial energy.

2 Preliminaries

In this part, we introduce some Lemmas and Corollary for the proof of our result.

Lemma 2.0.1. *[3, 4]. If $p:\Omega\to[1,\infty]$ is a measurable function θ on Ω and*

$$2<p_1\leq p(x)\leq p_2<\frac{2n}{n-2},\ n\geq 3. \tag{7}$$

Then, the embedding $H_0^1(\Omega)\hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact.

From the above lemma and by applying the Sobolev embedding theorem, we can derive the following corollary:

Corollary 1. *If $p:\Omega\to[1,\infty]$ is a measurable function θ on Ω and we give the sufficient conditions for $p(x)$ and $q(x)$*

$$2<p_1\leq p(x)\leq p_2<q_1\leq q(x)<q_2<\frac{2n}{n-2m} \tag{8}$$

Then, the embedding $H_0^m(\Omega)\hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact.

Lemma 2.0.2. *The energy associated with the problem (1) given by (2) satisfies the*

$$E'(t)=-\int_\Omega|\theta_t|^{p(x)}dx\leq 0 \tag{9}$$

and the inequality $E(t)\leq E(0)$ holds, where

$$\begin{aligned}E(0)&=\frac{1}{2}\|\theta_1\|^2+\frac{1}{2}\left\|\mathcal{P}^{\frac{1}{2}}\theta_0\right\|^2+\frac{1}{2(\gamma+1)}\left\|\mathcal{P}^{\frac{1}{2}}\theta_0\right\|^{2(\gamma+1)}\\&\quad+\int_\Omega\frac{1}{q(x)}|\theta_0|^{q(x)}\ln|\theta_0|\,dx+\int_\Omega\frac{1}{q^2(x)}|\theta_0|^{q(x)}dx.\end{aligned} \tag{10}$$

where

$$\begin{aligned}E(t)&=\frac{1}{2}\|\theta_t\|^2+\frac{1}{2}\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2+\frac{1}{2(\gamma+1)}\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^{2(\gamma+1)}\\&\quad+\int_\Omega\frac{1}{q^2(x)}|\theta|^{q(x)}dx-\int_\Omega\frac{|\theta|^{q(x)}}{q(x)}\ln|\theta|\,dx.\end{aligned} \tag{11}$$

Proof. We multiply the equation of (1) by θ_t, and integrating over Ω using integrating by parts, we get

$$E'(t)=-\int_\Omega|\theta_t|^{p(x)}dx\leq 0.$$

□

Lemma 2.0.3. *[10]. Let the conditions of (7) be fulfilled and let θ be the solution of (1). Then,*

$$\int_\Omega|\theta|^{q(x)}dx\geq\int_{\Omega_2}|\theta|^{q_1}dx=\|\theta\|_{q_1,\Omega_2}^{q_1} \tag{12}$$

where $\Omega_2=\{x\in\Omega:|\theta(x,t)|\geq 1\}$.

Lemma 2.0.4. *[10]. Under the assumptions stated in (8), the function $\mathcal{H}(t)$ provided above gives the following estimated:*

$$0<\mathcal{H}(0)\leq\mathcal{H}(t)\leq\frac{|\Omega|}{q_1e}+\frac{\mathcal{B}_s}{(s-q_2)q_1e}\|\nabla\theta\|_2^s,\ t\geq 0,$$

where s is chosen sufficiently small such that

$$\begin{cases} q_1 \leq q_2 < s < \infty, \text{ for } n = 1, 2, \\ q_1 \leq q_2 < s \leq \frac{2n}{n-2}, \text{ for } n \geq 3, \end{cases}$$

and $\mathcal{B}_s$ is a positive constant of embedding $H_0^1(\Omega)$ in $L^s(\Omega)$ such that

$$\|u\|_s \leq \mathcal{B}_s \|\nabla\theta\|_2, \ \forall\theta \in H_0^2(\Omega).$$

Where, $\mathcal{H}(t)$ is defined in (13).

From the above lemma and by applying the Sobolev embedding theorem, we can derive the following corollary:

Corollary 2. *Under the assumptions stated in (8), the function $\mathcal{H}(t)$ provided above gives the following estimated:*

$$0 < \mathcal{H}(0) \leq \mathcal{H}(t) \leq \frac{|\Omega|}{q_1 e} + \frac{\mathcal{B}_s}{(s - q_2) q_1 e} \left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^s, \ t \geq 0,$$

where s is chosen sufficiently small such that

$$\begin{cases} q_1 \leq q_2 < s < \infty, \text{ for } n \leq 2m, \\ q_1 \leq q_2 < s \leq \frac{2n}{n-2m}, \text{ for } n \geq 2m, \end{cases}$$

and $\mathcal{B}_s$ is a positive constant of embedding $H_0^m(\Omega)$ in $L^s(\Omega)$ such that

$$\|u\|_s \leq \mathcal{B}_s \left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2, \ \forall\theta \in H_0^m(\Omega).$$

3 Blow Up

In this part, we state and prove our main result.

Theorem 3.1. *Assume that (8) hold, and $E(0) < 0$. Then any solution of problem (1) blows up infinite time.*

Proof. Let

$$\mathcal{H}(t) = -E(t) \text{ for } t \geq 0, \tag{13}$$

since $E(t)$ is absolutely continuous, hence $\mathcal{H}'(t) \geq 0$ and

$$0 < \mathcal{H}(0) \leq \mathcal{H}(t) \leq \int_\Omega \frac{1}{q(x)} |\theta|^{q(x)} \ln|\theta| \, dx.$$

We define

$$\Phi(t) = \mathcal{H}^{1-\sigma}(t) + \varepsilon \int_\Omega \theta\theta_t dx, \tag{14}$$

with $\sigma > 0$ is small enough to be chosen later and such that

$$0 < \sigma \leq \min\left\{\frac{q_1 - 2}{2q_1}, \frac{q_1 - p_2}{q_1(p_2 - 1)}, \frac{2(q_1 - p_1)}{s(p_1 - 1) q_1}, \frac{2(q_1 - p_1)}{s(p_2 - 1) q_1}\right\}. \tag{15}$$

Differentiation of (14), and using (1) we get

$$\begin{aligned}\Phi'(t) &= (1-\sigma)\mathcal{H}^{-\sigma}(t)\mathcal{H}'(t) + \varepsilon\|\theta_t\|^2 - \varepsilon\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2 \\ &\quad -\varepsilon\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^{2(\gamma+1)} + \varepsilon\int_\Omega |\theta_t|^{q(x)} \ln\theta dx - \varepsilon\int_\Omega \theta|\theta_t|^{p(x)-2}\theta_t dx. \end{aligned} \tag{16}$$

Add and subtract $\varepsilon(1-\eta)q_1\mathcal{H}(t)$ with $0<\eta<\frac{q_1-2}{q_1}$ on the righthand side of (16), to arrive at

$$\begin{aligned}\Phi'(t) \geq\;& (1-\sigma)\mathcal{H}^{-\sigma}(t)\mathcal{H}'(t)+\varepsilon(1-\eta)q_1\mathcal{H}(t)+\varepsilon\left(1+\frac{(1-\eta)q_1}{2}\right)\|\theta_t\|^2\\ &+\varepsilon\left(\frac{(1-\eta)q_1}{2}-1\right)\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2+\varepsilon\left(\frac{(1-\eta)q_1}{2(\gamma+1)}-1\right)\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^{2(\gamma+1)}\\ &+\varepsilon\eta\int_\Omega|\theta|^{q(x)}\ln\theta dx+\varepsilon\left(\frac{(1-\eta)q_1}{q_2^2}\right)\int_\Omega|\theta|^{q(x)}dx-\varepsilon\int_\Omega\theta\theta_t|\theta_t|^{p(x)-2}dx\end{aligned}$$

taking into account

$$\frac{1}{q_2^2}\int_\Omega|\theta|^{q(x)}dx<\frac{1}{q_1}\int_\Omega|\theta_t|^{q(x)}\ln\theta dx,$$

we get

$$\begin{aligned}\Phi'(t) \geq\;& (1-\sigma)\mathcal{H}^{-\sigma}(t)\mathcal{H}'(t)+\varepsilon(1-\eta)q_1\mathcal{H}(t)+\varepsilon\left(1+\frac{(1-\eta)q_1}{2}\right)\|\theta_t\|^2\\ &+\varepsilon\left(\frac{(1-\eta)q_1}{2}-1\right)\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2+\varepsilon\left(\frac{(1-\eta)q_1}{2(\gamma+1)}-1\right)\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^{2(\gamma+1)}\\ &+\varepsilon\frac{q_1}{q_2^2}\int_\Omega|\theta|^{q(x)}dx-\varepsilon\int_\Omega\theta\theta_t|\theta_t|^{p(x)-2}dx.\end{aligned}\tag{17}$$

Combining (12), (17) result in

$$\begin{aligned}\Phi'(t) \geq\;& (1-\sigma)\mathcal{H}^{-\sigma}(t)\mathcal{H}'(t)+\varepsilon\beta\left[\mathcal{H}(t)+\|\theta_t\|^2+\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2+\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^{2(\gamma+1)}+\int_\Omega|\theta|^{q(x)}dx\right]\\ &-\varepsilon\int_\Omega\theta\theta_t|\theta_t|^{p(x)-2}dx\\ \geq\;& (1-\sigma)\mathcal{H}^{-\sigma}(t)\mathcal{H}'(t)+\varepsilon\beta\left[\mathcal{H}(t)+\|\theta_t\|^2+\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2+\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^{2(\gamma+1)}+\|\theta\|_{q_1,\Omega_2}^{q_1}\right]\\ &-\varepsilon\int_\Omega\theta\theta_t|\theta_t|^{p(x)-2}dx,\end{aligned}\tag{18}$$

where

$$\beta=\min\left\{(1-\eta)q_1,\left(1+\frac{(1-\eta)q_1}{2}\right),\left(\frac{(1-\eta)q_1}{2}-1\right),\left(\frac{(1-\eta)q_1}{2(\gamma+1)}-1\right),\frac{q_1}{q_2^2}\right\}.$$

Now, by applying Young's inequality, we can make an estimate for the last term in (16) as demonstrated below

$$\begin{aligned}\int_\Omega\theta\theta_t|\theta_t|^{p(x)-2}dx \leq\;& \frac{1}{p_1}\int_\Omega\gamma^{p(x)}|\theta|^{p(x)}dx\\ &+\frac{p_2-1}{p_2}\int_\Omega\gamma^{-\frac{p(x)}{p(x)-1}}|\theta_t|^{p(x)}dx,\ \ (\forall\gamma>0).\end{aligned}\tag{19}$$

As a result, by taking γ such that

$$\gamma^{-\frac{p(x)}{p(x)-1}}=k\mathcal{H}^{-\sigma}(t)\ \ k>0,$$

substituting the statement into equation (19) with a sufficiently large k to be specified later, we derive the following inequality:

$$\begin{aligned}\int_\Omega\theta|\theta_t|^{p(x)-1}dx \leq\;& \frac{1}{p_1}\int_\Omega k^{1-p(x)}\mathcal{H}^{\sigma(p(x)-1)}(t)|\theta|^{p(x)}dx\\ &+\frac{p_2-1}{p_2}k\mathcal{H}^{-\sigma}(t)\mathcal{H}'(t),\ \forall\gamma>0.\end{aligned}\tag{20}$$

The result of joining (18) with (20)

$$\begin{aligned}\Phi'(t) &\geq \left[(1-\sigma)-\varepsilon\frac{p_2-1}{p_2}k\right]\mathcal{H}^{-\sigma}(t)\,\mathcal{H}'(t)\\ &+\varepsilon\beta\left[\mathcal{H}(t)+\|\theta_t\|^2+\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2+\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^{2(\gamma+1)}+\|\theta\|_{q_1,\Omega_2}^{q_1}\right]\\ &-\varepsilon\frac{k^{1-p_1}}{p_1}\mathcal{H}^{\sigma(p_2-1)}(t)\int_\Omega|\theta|^{p(x)}\,dx. \end{aligned} \tag{21}$$

Using Corollary 2, we obtain

$$\begin{aligned}&\mathcal{H}^{\sigma(p_2-1)}(t)\int_\Omega|\theta|^{p(x)}\,dx\\ \leq\ & 2^{\sigma(p_2-1)-1}C\left(\frac{|\Omega|}{q_1e}\right)^{\sigma(p_2-1)}\left(\left(\|\theta\|_{q_1,\Omega_2}^{q_1}\right)^{\frac{p_1}{q_1}}+\left(\|\theta\|_{q_1,\Omega_2}^{q_1}\right)^{\frac{p_1}{q_1}}\right)\\ &+2^{\sigma(p_2-1)-1}C\frac{(\mathcal{B}_s)^{s\sigma(p_2-1)}}{(s-q_2)\,eq_1}\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^{s\sigma(p_2-1)}\left(\|\theta\|_{q_1,\Omega_2}^{p_1}+\|\theta\|_{q_1,\Omega_2}^{p_1}\right). \end{aligned} \tag{22}$$

We will estimate the terms to the right of (22) using Young's inequality, we get

$$\begin{aligned}\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^{s\sigma(p_2-1)}\|\theta\|_{q_1,\Omega_2}^{p_1} &\leq \frac{p_1}{q_1}\|\theta\|_{q_1,\Omega_2}^{q_1}+C\frac{q_1-p_1}{q_1}\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^{\frac{s\sigma(p_2-1)q_1}{q_1-p_1}}\\ &= \frac{p_1}{q_1}\|\theta\|_{q_1,\Omega_2}^{q_1}+C\frac{q_1-p_1}{q_1}\left(\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2\right)^{\frac{s\sigma(p_2-1)q_1}{2(q_1-p_1)}}, \end{aligned}$$

similarly

$$\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^{s\sigma(p_2-1)}\|\theta\|_{q_1,\Omega_2}^{p_2}\leq\frac{p_2}{q_1}\|\theta\|_{q_1,\Omega_2}^{q_1}+C\frac{q_1-p_2}{q_1}\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^{\frac{s\sigma(p_2-1)q_1}{q_1-p_1}}.$$

Using the following inequality

$$a^z\leq a+1\leq\left(1+\frac{1}{b}\right)(a+b);\ \forall a\geq 0,\ 0<z<1,\ b\geq 0, \tag{23}$$

and condition (8) with $a=\|\theta\|_{q_1,\Omega_2}^{q_1}$, $c_1=1+\frac{1}{\mathcal{H}(0)}$, $b=\mathcal{H}(0)$ and $z=\frac{p_1}{q_1}(z=\frac{p_2}{q_1})$, we get

$$\begin{aligned}\left(\|\theta\|_{q_1,\Omega_2}^{q_1}\right)^{\frac{p_1}{q_1}}+\left(\|\theta\|_{q_1,\Omega_2}^{q_1}\right)^{\frac{p_2}{q_1}} &\leq 2c_1\left(\|\theta\|_{q_1,\Omega_2}^{q_1}+\mathcal{H}(0)\right)\\ &\leq 2c_1\left(\|\theta\|_{q_1,\Omega_2}^{q_1}+\mathcal{H}(t)\right)\end{aligned}$$

and condition (15) with $a=\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^2$, $c_2=1+\frac{1}{H(0)}$, $b=\mathcal{H}(0)$ and $z=\frac{s\sigma(p_2-1)q_1}{2(q_1-p_1)}$, we have

$$\begin{aligned}\left(\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^2\right)^{\frac{s\sigma(p_2-1)q_1}{2(q_1-p_1)}} &\leq c_2\left(\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2+\mathcal{H}(0)\right)\\ &\leq c_2\left(\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2+\mathcal{H}(t)\right)\end{aligned}$$

also, $a=\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^2$, $c_3=1+\frac{1}{\mathcal{H}(0)}$, $b=\mathcal{H}(0)$ and $z=\frac{s\sigma(p_2-1)q_1}{2(q_1-p_2)}$, we obtain

$$\left(\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^2\right)^{\frac{s\sigma(p_2-1)q_1}{2(q_1-p_2)}}\leq c_3\left(\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2+\mathcal{H}(t)\right)$$

and so, (22)

$$\mathcal{H}^{\sigma(p_2-1)}(t)\int_\Omega |\theta|^{p(x)}\,dx \le C\left(\|\theta\|_{q_1,\Omega_2}^{q_1}+\mathcal{H}(t)+\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2\right),\ \forall t\in[0,T], \tag{24}$$

where $C=C(\Omega,\ e,\ a,\ p_{1,2},\ q_{1,2})>0$. Combining (21) and (24), we get

$$\begin{aligned}\Phi'(t) &\ge \left[(1-\sigma)-\varepsilon\frac{p_2-1}{p_2}k\right]\mathcal{H}^{-\sigma}(t)\,\mathcal{H}'(t)\\ &+\varepsilon\left[\beta-\frac{k^{p_2-1}}{p_2}C\right]\left[\mathcal{H}(t)+\|\theta_t\|^2+\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2+\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^{2(\gamma+1)}+\|\theta\|_{q_1,\Omega_2}^{q_1}\right].\end{aligned} \tag{25}$$

At this point we pick $\gamma=\beta-\frac{k^{p_2-1}}{p_2}C\ge 0$, (it is the case when $k>\left(\frac{\beta p_1}{C}\right)^{\frac{1}{1-p_1}}$. Once k is fixed we pick $\varepsilon>0$ sufficient small so that

$$(1-\sigma)-\varepsilon\frac{p_2-1}{p_2}k\ge 0$$

and

$$\Phi(0)=\mathcal{H}^{1-\sigma}(0)+\varepsilon\int_\Omega \theta_0(x)\,\theta_1(x)\,dx>0.$$

Hence (25) takes the form

$$\Phi'(t)\ge\gamma\left[\mathcal{H}(t)+\|\theta_t\|^2+\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|^2+\left\|\mathcal{P}^{\frac{1}{2}}\theta\right\|_2^{2(\gamma+1)}+\|\theta\|_{q_1,\Omega_2}^{q_1}\right]. \tag{26}$$

Therefore, we have

$$\Phi(t)\ge\Phi(0)>0,\ \text{for all } t\ge 0$$

On the other hand from (14),

$$\Phi^{\frac{1}{1-\sigma}}(t)\le 2^{\frac{1}{(1-\sigma)}}\left(\mathcal{H}(t)+\left|\int_\Omega\theta\theta_t dx\right|^{\frac{1}{(1-\sigma)}}\right) \tag{27}$$

by utilizing Hölder's inequality, it becomes

$$\begin{aligned}\left|\int_\Omega\theta\theta_t dx\right|^{\frac{1}{(1-\sigma)}} &\le C\|\theta\|_{q_1}\|\theta_t\|_2\\ &\le C\|\theta\|_{q_1,\Omega}\|\theta_t\|_2.\end{aligned}$$

Again, algebraic inequality (23), with $a=\|\theta\|_{q_1,\Omega_2}^{q_1}$, $c=1+\frac{1}{\mathcal{H}(0)}$, $b=\mathcal{H}(0)$ and $0<z=\frac{2p_1}{(1-2\alpha)q_1}\le 1$ (see 15), gives

$$\left(\|\theta\|_{q_1,\Omega_2}^{q_1}\right)^{\frac{2}{(1-2\sigma)q_1}}\le C\left(\|\theta\|_{q_1,\Omega_2}^{q_1}+\mathcal{H}(t)\right).$$

Thus, Young's inequality gives

$$\begin{aligned}\left|\int_\Omega\theta\theta_t dx\right|^{\frac{1}{(1-\sigma)}} &\le C\left[\|\theta\|_{q_1,\Omega_2}^{\frac{2(1-\sigma)}{1-2\sigma}}+\|\theta_t\|_2^{2(1-\sigma)}\right]^{\frac{1}{(1-\sigma)}},\\ &\le C\left[\left(\|\theta\|_{q_1,\Omega_2}^{q_1}\right)^{\frac{2}{(1-2\sigma)q_1}}+\|\theta_t\|_2^2\right],\\ &\le C\left[\|\theta\|_{q_1,\Omega_2}^{q_1}+\mathcal{H}(t)+\|\theta_t\|_2^2\right],\ \text{for all } t\ge 0,\end{aligned}$$

joining it with (26) and (27) yields

$$\Phi'(t)\ge\zeta\Phi^{\frac{1}{1-\sigma}}(t) \tag{28}$$

where $\zeta = \zeta(\varepsilon, \gamma, C) > 0$. By taking a simple integration of (28) over $(0, t)$ we deduce that

$$\Phi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Phi^{\frac{\sigma}{1-\sigma}}(0) - \frac{\sigma}{1-\sigma}\zeta t}. \tag{29}$$

Consequently, $\Phi(t)$ blows up in a finite time T^*

$$T^* \leq \frac{1-\sigma}{\zeta\sigma\Phi^{\frac{\sigma}{1-\sigma}}(0)}.$$

□

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